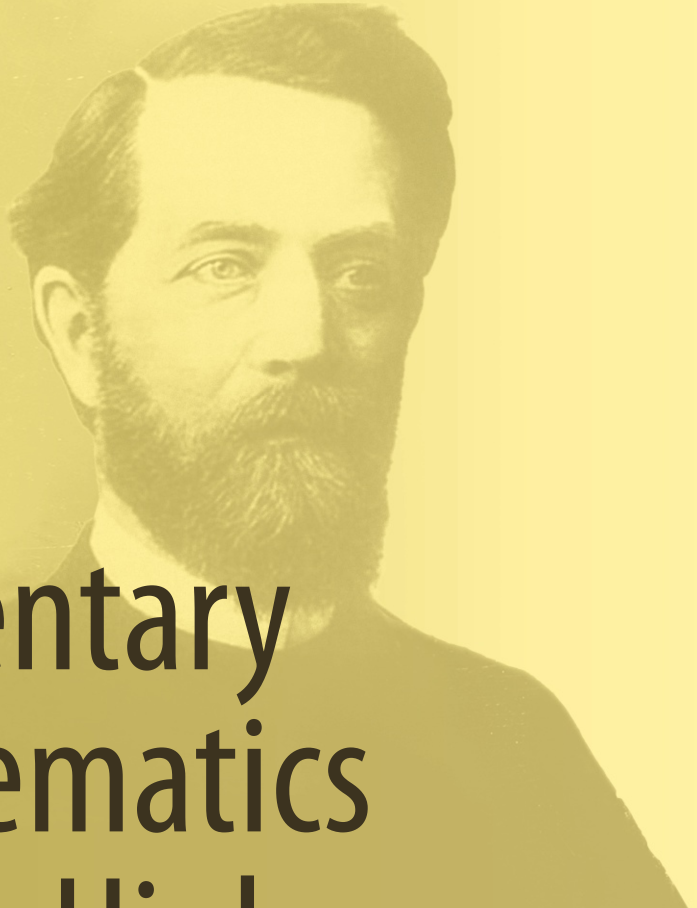

Felix Klein

Elementary Mathematics from a Higher Standpoint

Volume II: Geometry

Springer

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Preface to the 2016 Edition

In general, for the preface for this volume II, I can refer to the preface for volume I – especially regarding the necessary correction of „advanced“ to „higher“, regarding the notion of *elementary mathematics* and of *elementarisation*, and regarding the need for a revised translation.

I should like to highlight the special features of this volume. While the first volume is dedicated to the analytic side of mathematics, this volume complements it by exposing geometry. But the second volume is complementary to the first one in two more respects. It is characteristic for the first volume that Klein – in treating arithmetic, algebra, and analysis – had always emphasised a geometric approach to the concepts: to provide a geometric interpretation and to reveal the key function of *Anschauung* in developing and in understanding the analytic concepts. Here, for geometry, Klein carefully elaborates the analytic side of the geometric concepts. His major aim is to show the ultimate unity of mathematics.

The second complimentary function is Klein's masterful manner of elaborating the process of elementarisation for the whole of geometry. In fact, it was his key scientific achievement to have changed the character of geometry: until his times, there existed side by side a number of – one might say – different geometries, having been established by the one or the other mathematician, for some specific objective and continuing in a rather dispersed manner and without seeking mutual relations or developments. Klein's *Erlanger Programm* of 1872 is emblematic for the radical change of this situation, for having succeeded in a new, "higher" unity of geometry. All the enormous variety of geometrical theories, approaches and subdisciplines had become unified – thanks to rebuilt foundations and, consequently, new "elements". Particularly illuminating is to see how the development of non-Euclidean geometries became a part of the processes leading to this unified architecture. As Klein has put it, based on Cayley: "Projective geometry is all geometry". Evidently, this elementarisation provided an exemplary methodology for realising a form of mathematics teacher training, which provided future teachers with a proper standpoint for their action later in their profession.

The structure of the second volume is different from the first one. While issues of teaching were there always integrated into the various conceptual topics, Klein

decided to give the conceptual exposition as a coherent presentation and to discuss questions of teaching in a separate chapter. In this manner, Klein was able to realise this illuminating way of revealing the essential unity of geometry and its restructured elements. Evidently, this last chapter on teaching was very dear to Klein and an integral part of his conception. It remains therefore absolutely incomprehensible why the two American translators, Earle Raymond Hedrick and Charles Albert Noble, decided in 1939 to omit this chapter – and without any notice.

This chapter analyses the teaching of geometry in various European countries, first at the time of the first international reform movement, and then for the interwar period, which is otherwise not well explored. One gets revealing insight in the characteristic differences of methodology and organisation of school mathematics and of geometry teaching. Moreover, a number of issues of teaching geometry are discussed.

Regarding the first translation, their second volume reveals analogous problems mentioned in my preface to the first volume: wrong or inconsistent mathematical terminology and misunderstandings of the German text. To give just one example: on p. [4], the German text speaks, in a geometric context, of "Inhaltsbestimmung". Well, "Bestimmung" is "determination", but "Inhalt"? When no context is given, there are two meanings for this German term: either "area" (or "volume") or "content". Although the context is clearly geometric, they translated with "content". Even more strangely, they continue at first with "area" – but after a few pages, they again use "content".

In this volume, too, the reader will find, again in square brackets and in bold, *the page numbering of the original edition*. Cross-references in notes and in the text *refer to this numbering*, as well as *the name index and the subject index* (that is, the original text has not been changed in this respect).

In the present translation I have added, when possible, the first names of the persons mentioned. In the German edition, as it was customary at that time, the first names were indicated only with the initials. The bibliographic references in the notes have also been completed, when needed.

In the notes of English version of 1939, Hedrick and Noble had sometimes added references for recent pertinent American publications; these have been maintained. Several additional notes have been introduced; they are marked by square brackets.

As in volume I, the German names of the nine grades of secondary schools have been maintained, for greater exactness: *Sexta*, *Quinta*, *Quarta*, *Unter-Tertia*, *Ober-Tertia*, *Unter-Sekunda*, *Ober-Sekunda*, *Unter-Prima*, *Ober-Prima*.

I am thanking Leo Rogers for his careful re-reading of the book, and the various colleagues whom I asked advice, in particular Geoffrey Howson.

We are grateful to Dover Publication to have authorised the use of their book "Elementary Mathematics from an Advanced Standpoint", translated by E. R. Hedrick and C. A. Noble, for a revised new edition.

Gert Schubring

Preface to the First Edition

In the preface to Part I of these lecture notes (Arithmetic, Algebra, Analysis) I expressed a doubt as to whether Part II, devoted to geometry, could appear soon. Nevertheless it has been possible to complete it, thanks to the diligence of Mr. Hellinger.

Concerning the origin and purpose of this series of lecture courses I have nothing especial to add to what was said in the foreword to Part I. However, a comment seems necessary concerning the new form, which this second part has assumed.

This form is, in fact, quite unlike that of Part I. I made up my mind to give, above all, a *comprehensive view* of the field of geometry, of such a range as I should wish every teacher in a secondary school would master; the discussions about geometry *teaching* were pushed into the background and were placed in connected form at the end, insofar as there was room left, but now in a connected form.

The choice of this new order was motivated partly by the desire to avoid a stereotyped form. There were, however, more important and deeper reasons. In geometry we possess no such homogeneous textbooks corresponding to the general level of the science, such as they exist in algebra and analysis, thanks to the prototype of the French *cours*. We find, rather, one aspect treated here, another particular aspect there, of this extensive subject, just as it has been developed by one or another group of researchers. In contrast to this, it seemed to be demanded by the pedagogic and the general scientific purposes, which I am intending that I attempt a more unified presentation.

I close with the wish that the two complementary parts of my *Elementary Mathematics from a Higher Standpoint* which are herewith completed may find the same friendly reception in the teaching world as the lectures on the organisation of mathematics teaching by Mr. Schimmack and myself, which appeared last year.

Göttingen, Christmas, 1908 **Klein**

Preface to the Third Edition

In virtue of the overall plan for the new edition of my lithographed lectures, which I explained in the preface to the third edition of the first volume, the text and presentation of the present second volume, have remained unaltered, except for small changes in detail and a few insertions.[1]

The two supplements, which concern literature about scientific and pedagogic aspects, which was not considered in the original text, were prepared by Mr. Seyfarth, after repeated conferences with me. He assumed again the major portion of the burden entailed by the publication. Messrs. Ernst Hellinger, H. Vermeil, and Alwin Walther assisted him in the proof reading. Mr. Vermeil undertook the preparation of the two indexes. I am obliged to these gentlemen, and also to the publisher Julius Springer, who showed at all occasions much cooperation in realising my proposals.

Göttingen, May, 1925 **Klein**

[1] Newly added remarks are indicated by square brackets.

Contents

Introduction [1]

Aim and Form of this Lecture Course

Gentlemen! The lecture course, which I now begin, will be an immediate continuation of, and a supplement to, my course of last winter.[2] My purpose now, as it was then, is to summarise all the mathematics that you studied during your student years, insofar as this could be of interest for the future teacher, and, in particular, to show its importance for the practice of school teaching. I carried out this plan, during the winter semester, for *Arithmetic*, *Algebra*, and *Analysis*. During the current semester, attention will be given to *geometry*, which was then left aside. In this lecture course, comprehension of our considerations will be independent of knowledge of the preceding lecture course. Moreover, I shall give the whole a somewhat different tone: In the foreground I shall place, let me say, the *encyclopaedic approach* – you will be offered a *survey of the entire field of geometry* into which you can arrange, as into a rigid frame, all the separate items of knowledge which you have acquired in the course of your study, in order to have them at hand when occasion to apply them arises. Only afterward shall that *interest in mathematics teaching* appear by itself, which was always my emphasis last winter.

I should like to refer to a *vacation course* for teachers of mathematics and physics, which was given here in Göttingen during the Easter vacation in 1908. In it I gave an account of my winter lecture course. In connection with this, and also with the talk by Professor *Otto Behrendsen* of the local Gymnasium, there arose an interesting and stimulating discussion concerning the reorganisation of teaching arithmetic, algebra, and analysis, and more particularly about the introduction of differential and integral calculus into the schools.[3] The participants showed an extremely gratifying interest in these questions and, in general, in our efforts to bring the university into living touch with the schools. I hope that my present lecture [2]
course also may exert an influence in this direction. May they contribute their part toward the elimination of the old complaint, which we have had to hear continu-

[2] [Appeared as Volume I of this series of lecture notes on *Elementary Mathematics from a Higher Standpoint*, Berlin, 1924, 3rd edition. The quotation "Part I" refers to the third edition.]

[3] See the report by Rudolf Schimmack, *Ueber die Gestaltung des mathematischen Unterrichts im Sinne der neueren Reformideen*, Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, vol. 39 (1908), pp. 513–527, (also printed separately, Leipzig, 1908).

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
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ally – and often justly – from the schools: higher education provides, indeed, much of a special nature, but it leaves the beginning teacher entirely without orientation as to many important general things which he could really use later.

Concerning now the *topics of this lecture course*, let me say that, as in the preceding course, I shall now and then have to presuppose knowledge of important theorems from all of the fields of mathematics, which you have studied, in order to lay emphasis upon a *general survey of the whole*. To be sure, I shall always try to assist your memory by brief statements, so that you can easily orient yourself in the literature. On the other hand, I shall draw attention, more than is usually done, and as I did in Part I, to the *historical development of the science*, to the accomplishments of its great pioneers. I hope, by discussions of this sort, to further, as I like to say, your *general mathematical culture*: alongside of knowledge of details, as these are supplied by the special lecture courses, there should be a grasp of subject-matter and of historical contexts.

The Efforts for "Fusion"

Allow me to make a last general remark, in order to avoid a misunderstanding, which might arise from the nominal separation of this "geometric" part of my lectures from the first arithmetic part. In spite of this separation, I advocate here, as always in such general lecture courses, a tendency which I like best to designate by the catchphrase "*fusion of arithmetic and geometry*" – meaning by arithmetic, as is usual in the schools, the field which includes not merely the theory of integers, but also the whole of algebra and analysis. Some are inclined, especially in Italy, to use the word "*fusion*" as a catchphrase for efforts, which are restricted to geometry. In fact, it has long been the custom in secondary as well as in higher education, first to study geometry of the plane and then, entirely separated from it, the geometry of space. On this account, space geometry is unfortunately often slighted, and the noble faculty of space intuition, which we possess originally, is stunted. In contrast to this, the "fusionists" wish to treat the plane and space together, in order not to restrict our thinking artificially to two dimensions. This endeavour also meets my approval, but I am thinking, at the same time, of a still more far-reaching fusion. Last semester I endeavoured always to enliven the abstract discussions of arithmetic, algebra, and analysis by means of figures and graphic methods, which
[3] bring the things nearer to the individual and often only thus succeed in making him understand, for the first time, why he should be interested in them. Similarly, I shall now, from the very beginning, accompany space intuition, which, of course, will hold first place, with analytic formulas, which facilitate in the highest degree the precise formulation of geometric facts.

You will most easily see what I am meaning when I turn now to our subject; at first a series of simple geometric fundamental forms will be considered.

First Part: The Simplest Geometric Formations

I. Line segment, Area, Volume as Relative Quantities

Definition by Determinants; Interpretation of Signs

You will notice by the heading of this section that I am following the intention announced above, of examining simultaneously the corresponding magnitudes on the straight line, in the plane, and in space. At the same time, however, we shall take into account the principle of fusion by making use at once of the *rectangular system of coordinates* for the purpose of analytic formulation.

If we have a *line segment*, let us think of it as laid upon the x-axis. If the abscissas of its endpoints are x_1 and x_2, its *length* is $x_1 - x_2$, and we may write this difference in the form of the determinant

$$(1,2) = x_1 - x_2 = \frac{1}{1}\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}.$$

Similarly, the *area of a triangle* in the x-y-plane which is formed by the three points 1, 2, 3, with coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), will be

$$(1,2,3) = \frac{1}{1\cdot 2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Finally, we have, for the *volume of the tetrahedron* made by the four points 1, 2, 3, 4, with coordinates $(x_1, y_1, z_1), \ldots, (x_4, y_4, z_4)$, the formula

$$(1,2,3,4) = \frac{1}{1\cdot 2\cdot 3}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

We say ordinarily that the length, or, as the case may be, the area or the volume, **[4]** is equal to the *absolute value* of these several magnitudes, whereas, actually, our formulas furnish, over and above that, a *definite sign*, which depends upon the order in which the points are taken. We shall make it a fundamental rule always

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
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to take into account those signs, which the analytic formulas supply, in geometry. We must accordingly inquire as to the *geometric significance of the sign in these determinations of areas*.

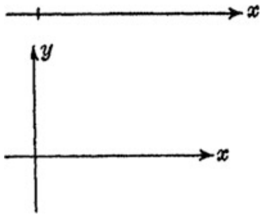

Figure 1

It is important, therefore, how we choose the *system of rectangular coordinates*. Let us, then, at the outset, adopt a convention, which is, of course, arbitrary, but which must be binding in all cases. In the case of *one dimension*, we shall think of the positive x-axis as always pointing to the right. In the plane, the positive x-axis will be directed toward the right, the positive y-axis upward (see Fig. 1). If we were to let the y-axis point downward, we should have an essentially different coordinate system, one which would be a reflection of the first and not superimposable upon it by mere motion in the plane, i.e., without extending into space. Finally, the *coordinate system in space* will be obtained from the one in the plane by adding to the latter a z-axis directed positively to the *front* (see Fig. 2). A choice of the z-axis pointing positively to the rear would give, again, an essentially different coordinate system, one which could not be made to coincide with ours by any movement in space.[4]

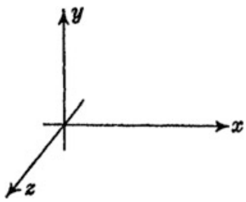

Figure 2

If we always adhere to these conventions, we shall find the *interpretation of our signs in simple geometric properties of the succession of points as these are determined by their numbering*.

[4] These two systems are distinguished as "right-handed" and "left-handed" because they correspond respectively to the position of the first three fingers of the right and left hand. (See Vol. I, p. [70])

For the *segment* (1, 2) this property is obvious: *The expression* $x_1 - x_2$ *for its length is positive or negative according as point* 1 *lies to the right or to the left of point* 2.

In the case of the *triangle*, we obtain: *The formula for the area has the positive or the negative sign according as passing around the triangle from the vertex* 1 *to* 3 *via 2 turns out to be counterclockwise or the reverse*. We shall prove this by taking, first, a conveniently placed special triangle, calculating directly the determinant, which expresses its area, and then, through an argument about continuity, resolve **[5]**
the general case. We consider that triangle which has, as its first vertex, the unit point on the x-axis ($x_1 = 1$, $y_1 = 0$), as its second the unit point on the y-axis ($x_2 = 0$, $y_2 = 1$), and as its third the origin ($x_3 = 0$, $y_3 = 0$). According to our convention about the system of coordinates, we must pass around this triangle in the counterclockwise sense (see Fig. 3), and our formula for its area yields the positive value:

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = +\frac{1}{2}\,.$$

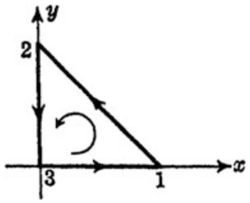

Figure 3

Now we can bring the vertices of this triangle, by continuous deformation, into coincidence with those of any other triangle travelled around in the same sense, and we can do this in such a way that the three vertices of the triangle shall at no time become collinear. In this process, our determinant changes its value continuously, and since it vanishes only when the points 1, 2, 3 are collinear, it must always remain positive. This establishes the fact that the area of any triangle whose boundary is travelled around in counterclockwise sense is positive. If we interchange two vertices of the original triangle, we see at once that every triangle, which is travelled around *in clockwise sense* has a negative area.

We can now treat the *tetrahedron* in analogous fashion. We start, again, with a conveniently placed tetrahedron. As first, second, and third vertices, we choose, in order, the unit points on the x-, y-, and z-axes, and as fourth vertex the origin (see Fig. 4). Its volume is therefore

$$\frac{1}{6}\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = +\frac{1}{6}\,.$$

It follows, as before, that every tetrahedron, which can be obtained from this one by continuous deformation while the four vertices never become co-planar (i.e., during which the determinant never vanishes), has positive volume. But one can characterise all these tetrahedrons by the sense in which that face-triangle $(2, 3, 4)$ is travelled around when it is looked at from the vertex 1. In this way we obtain the result: *The volume of the tetrahedron* $(1, 2, 3, 4)$ *which our formula yields is positive if the vertices* $2, 3, 4$, *looked at from vertex* 1, *follow one another in counterclockwise sense*; *otherwise it is negative*.

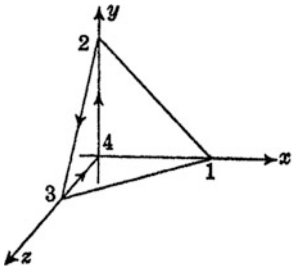

Figure 4

[6] We have thus, from our analytic formulas, actually deduced geometric rules which permit us to assign a definite sign to any segment, any triangle, any tetrahedron, if the vertices are given in a definite order. Great advantages are thus gained over the ordinary elementary geometry, which considers length and area as absolute magnitudes. Indeed, we can establish general simple theorems even there where elementary geometry must distinguish numerous cases according to the particular form of the figure.

Simple Applications; in Particular the Cross-Ratio

Let me begin with a very primitive example, the *ratio of the segments* made by three points on a line, say the x-axis. Denoting the three points by 1, 2, and 4 (see Fig. 5), as is the most convenient in view of what is to follow, we see that the ratio in question will be given by the formula

$$S = \frac{x_1 - x_2}{x_1 - x_4},$$

and it is clear that this quotient is positive or negative according as the point 1 lies outside or inside the segment (2, 4). If one gives, as is customary in elementary expositions, only the absolute value

$$|S| = \frac{|x_1 - x_2|}{|x_1 - x_4|},$$

we must always either refer expressly to the figure, or state in words whether we have in mind an inside or an outside point, which is, of course, more cumbersome. *The introduction of the sign thus takes account of the different possible* o r d e r s *of the points on the line*, a fact to which we shall often have to refer during this lecture course.

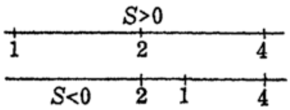

Figure 5

If we now add a fourth point 3, we can set up the *cross ratio* of the four points, that is,

$$D = \frac{x_1 - x_2}{x_1 - x_4} : \frac{x_3 - x_2}{x_3 - x_4} = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)}.$$

This expression has again a definite sign, and we see at once that $D < 0$ when the pair of points 1 and 3, on the one hand, and the pair 2 and 4, on the other hand, mutually separate one another; and that $D > 0$ in the opposite case, i.e., when 1 and 3 lie both outside or both inside the segment 2, 4. (See Figs. 6 and 7.) Thus there are always two essentially different positions, which yield the same absolute value D. If this absolute value alone is given, we must, moreover, give expressly the determination of the position. For example, if one defines harmonic points by the equation $D = 1$, as is still the custom, unfortunately, in the schools, one must include in the definition the demand of a separate position of the two pairs of points, whereas in our plan the *one* demand $D = -1$ is sufficient. **[7]**

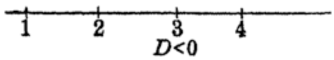

Figure 6

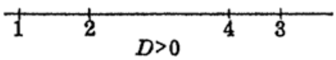

Figure 7

This practice of taking account of the sign is especially useful in *projective* geometry, in which, as you know, the cross-ratio plays a leading role. There we have the familiar theorem that four points on a line have the same cross-ratio as the four points, which arise when we project the given points from an arbitrary centre upon another line (perspective). If we now consider the cross-ratio as a relative magnitude, affected by a sign, the converse of this theorem holds without exception: If each of two sets of four points lies on one of two lines, and if they have the same

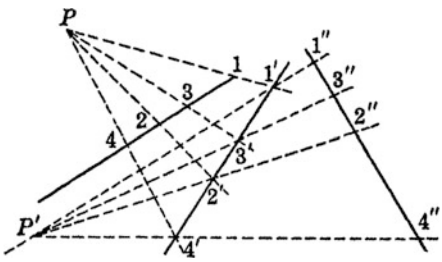

Figure 8

cross-ratio, they can be derived one from the other by projection, either single or repeated. For example, in Fig. 8, the sets 1, 2, 3, 4, and 1″, 2″, 3″, 4″ by projection from the centres P and P'. If, however, one knows only the absolute value of D, the corresponding theorem does not hold in this simple form; we should have to make a special assumption about the position of the points.

Area of Rectilinear Polygons

We have a more fruitful field if we proceed to *applications of our triangle formula*. Let us first select somehow a point 0 in the interior of a triangle (1, 2, 3) and let us join 0 to each of the vertices (see Fig. 9). Then the sum of the areas, thought of in the elementary sense as absolute magnitudes, of the three partial triangles is equal to the area of the original triangle. Thus we may write

$$|(1,2,3)| = |(0,2,3)| + |(0,3,1)| + |(0,1,2)|\,.$$

Given the positions in the figure, the order of the vertices, in all the triangles, as they appear in the above equation, is counterclockwise. Hence the areas $(1,2,3)$,
[8] $(0,2,3)$, $(0,3,1)$, $(0,1,2)$ – signed in the sense of our general definition – are all positive so that we may write our formula in the form

$$(1,2,3) = (0,2,3) + (0,3,1) + (0,1,2)\,.$$

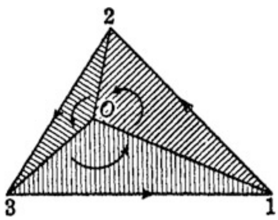

Figure 9

Now I assert *that the same formula also holds when* 0 *lies outside the triangle*, *and*, *further*, *when* 0, 1, 2, 3 *are any four points whatever in the plane*. If we take the position of Fig. 10, for example, we see that the boundaries of $(0, 2, 3)$ and $(0, 3, 1)$ are travelled around in a counterclockwise sense, but that of $(0, 1, 2)$ is in the clockwise sense, so that our formula for the areas, calculated as absolute quantities, would give

$$|(1, 2, 3)| = |(0, 2, 3)| + |(0, 3, 1)| - |(0, 1, 2)| .$$

The figure verifies the correctness of this equation.

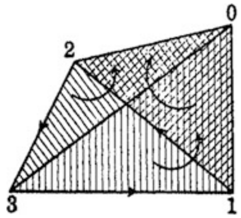

Figure 10

We shall give a general proof of our theorem by means of the *analytic definition*, whereby we shall recognise in our formula a well-known theorem of algebra, respectively of the theory of determinants. For convenience, let us take the point *0* as our origin $x = 0$, $y = 0$, which is obviously no essential specialization, and let us substitute for each of the four triangle areas the appropriate determinant. Then, omitting everywhere the factor $\frac{1}{2}$, it is left to prove that, for arbitrary values $x_1, \ldots, y_3$, the following relation holds:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} .$$

The value of each of the determinants on the right will remain unchanged if we replace the second and third 1 of the last column by zeros, since these elements enter only those minors, which are multiplied by zero, when we expand according to the top row. If we now make a cyclic interchange of rows in the last two determinants, which is permissible in determinants of the third, or, in fact, of any odd order, we can write our equation in the following form:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & 0 \\ 0 & 0 & 1 \\ x_3 & y_3 & 0 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} .$$

But this is an identity, for on the right there are only the minors of the last column of the left determinant, so that we have merely the well-known expansion of this

determinant according to the elements of a column. Thus, at one stroke, we have proved our theorem for all possible positions of the four points.

[9] We can now generalise this formula so that it will give the *area of any polygon*. Imagine that you had, say, the following problem in surveying: To determine the area of a rectilinear field after having measured the coordinates of the vertices $1, 2, \ldots, n-1, n$ (see Fig. 11). One who is not accustomed to operate with signs would then sketch the shape of the polygon, divide it up into triangles by drawing diagonals, perhaps, and then according to the particular shape of the field, paying especial regard to whether some of the angles are re-entrant, find the area as the sum or difference of the areas of the partial triangles. However, we can give at once a general formula, which will give the correct result quite mechanically without any necessity of looking at the figure: If 0 is any point in the plane, say the origin, then the area of our polygon, being travelled around in the sense $1, 2, \ldots, n$, will be

$$(1, 2, 3, \ldots, n) = (0, 1, 2) + (0, 2, 3) + \cdots + (0, n-1, n) + (0, n, 1),$$

whereby each triangle is to be taken with the sign determined by the sense in which the circuit about it is made. *The formula yields the area of the polygon positively or negatively according as the circuit of the polygon in the sense* $1, 2, \ldots, n$ *is counterclockwise or not.* It will suffice to write this formula. You yourselves can easily supply the proof.

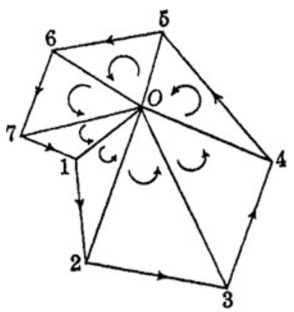

Figure 11

Instead of pursuing this example further, I prefer to take up some especially interesting cases, which, to be sure, could not arise in surveying, namely, cases of *polygons which are twisted upon themselves* as in the adjoining quadrilateral (see Fig. 12). If we wish here to talk at all about a definite area, it can only be the value which our formula yields. Let us consider what this value means geometrically. At the outset we notice that this must be independent of the particular location of the point 0. Let us place 0, as conveniently as possible, at the point where the twisting occurs. Then the triangles $(0, 1, 2)$ and $(0, 3, 4)$ will be zero and there remains:

$$(1, 2, 3, 4) = (0, 2, 3) + (0, 4, 1).$$

The first triangle has negative area, the second positive area; hence the area of our quadrilateral, if we ascribe it a circuit in the sense $(1, 2, 3, 4)$, is equal to the absolute **[10]**
value of the area of the part $(0, 4, 1)$ that was travelled around in counterclockwise sense, *diminished* by that of the part $(0, 2, 3)$ that was travelled around in a clockwise sense.

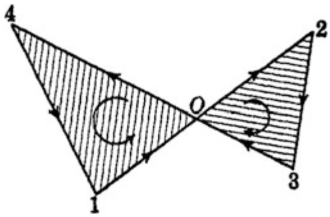

Figure 12

As a second example, let us examine the adjoined *star pentagon* (see Fig. 13). If we take O in the middle part, all the partial triangles in the sum

$$(0, 1, 2) + (0, 2, 3) + \cdots + (0, 5, 1)$$

are travelled in the positive sense; their sum covers the kernel, having five vertices, of the figure twice, and each of the five tips once. If we again compare a circuit around our polygon, done one-time along (1, 2, 3, 4, 5, 1), we see that every part of the boundary is travelled around counterclockwise and that, namely, we have the portion of the polygon which is doubly counted for determining the area, will be travelled twice around, but only once for the portion, which has to be counted only once.

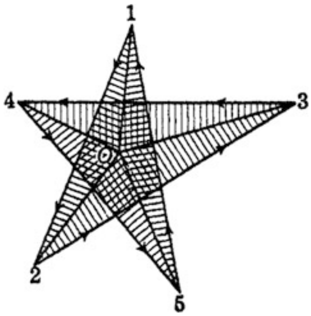

Figure 13

From these two examples we can infer the following *general rule*: *For any rectilinear polygon with* an *arbitrary number of twistings*, *our formula yields*, *as total area*, *the algebraic sum of the separate partial areas bounded by the polygonal path*, *whereby each of these partial areas is counted as often as we travel around*

its boundary when the circuit $(1, 2, 3, \ldots, n, 1)$ *is made once*, *this counting to be made positively or negatively according as we travel around the partial area in counterclockwise or clockwise sense*. You will have no difficulty in establishing the truth of this general theorem. The more I am recommending you to entirely appropriate these interesting area formulas by some examples.

Areas with Curvilinear Boundaries

Let us now pass from polygons to *areas with curvilinear boundaries*. We shall consider any closed curve whatever, which may twist upon itself any number of times. We assign a *definite sense of direction along this curve* and ask for the area bounded by the curve. We find this area in a natural manner if we approximate the curve by polygons having an increasing number of shorter and shorter sides (see Fig. 14) and calculate the limit of the areas of these polygons, found in the way we have just described. If

$$P(x, y) \quad \text{and} \quad P_1(x + dx, y + dy)$$

are two neighbouring vertices of such an approximating polygon, then its area consists of a sum of elementary triangles (OPP_1), that is of summands:

[11]

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ x + dx & y + dy & 1 \end{vmatrix} = \frac{1}{2}(x\,dy - y\,dx)\,.$$

In the limit, this sum becomes the line integral

$$\frac{1}{2}\int (x\,dy - y\,dx)$$

taken along the curve in the given direction, which, therefore, defines the area bounded by the curve. If we wish to interpret this definition geometrically, we can transfer right away to the new case the result just given for polygons: *Each*

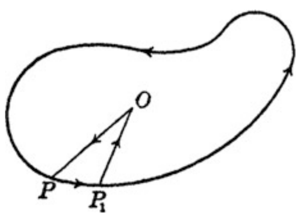

Figure 14

partial area enclosed by the curve is counted positively as many times as it is travelled around in a *counter-clockwise sense and negatively as many times as this is done in* a *clockwise sense while the given curve is traversed once in the prescribed sense*. For a simple curve, such as that of Fig. 14, the integral yields, accordingly, the exact area bounded by the curve, taken positively. In Fig. 15, the outer part is counted once positively, the inner part twice; in Fig. 16, the left-hand part is negative, while the right-hand part is positive, so that, altogether, a negative area results; in Fig. 17, one part is not counted at all, since it is encircled once positively and once negatively. Of course, curves can arise which, in this sense, bound a zero area. We obtain such a curve if we take the curve in Fig. 16 symmetric with respect to the point of twisting. Such a case presents nothing absurd when we recall that our determination of area rests upon a convenient assumption.

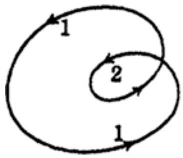

Figure 15

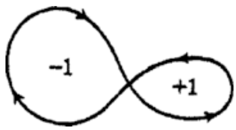

Figure 16

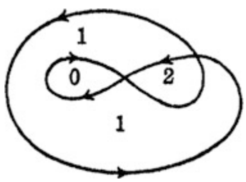

Figure 17

I shall now show you how appropriate these definitions are by considering the

Amsler's Polar Planimeter

This highly ingenious apparatus, very useful in practice, constructed in 1854 by the mechanic *Jacob Amsler* of Schaffhausen, effects the determination of areas pre-

cisely in the sense of our discussion above. Let me consider, first, the *theoretical basis of the construction*.

We think of a rod A_1A_2 (see Fig. 18) of length l moved in the plane in such
[12] a way that A_1 and A_2 describe separate closed curves and the rod itself returns to its initial position. We wish to find the area, which the rod sweeps over, counting the several parts of this area as positive or negative, according as they are swept over in one sense or in the other. To this end, we replace- according to the limit process to be realised for any integration – the continuous motion of the rod by a succession of arbitrarily small jerkily "elementary motions" from one position $1\,2$ to a neighbouring one $1'\,2'$. The actual area swept out by the rod will be the limit of the sum of all the "elementary quadrilaterals" $(1, 1', 2', 2)$ described during these elementary motions, and it is easy to see that the sense of the motion of the rod is taken into account properly if we give to each elementary quadrilateral the sign corresponding to a circuit in the sense $1, 1', 2', 2$. Now we can compose each elementary motion of the rod A_1A_2 from three steps (see Fig. 19):

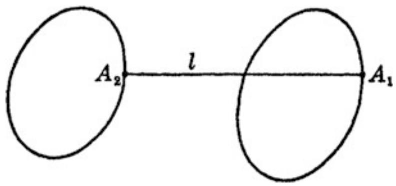

Figure 18

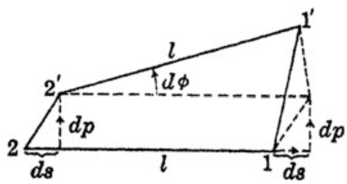

Figure 19

(1) A translation in the direction of the rod by an amount ds.
(2) A translation *normal* to its direction by an amount dp.
(3) A rotation about the end A_2 through an angle $d\phi$.

In this way the areas $0 \cdot ds$, $l \cdot dp$, $(l^2/2)d\phi$, respectively, will be swept out. We can replace the area of the elementary quadrilateral by the sum of these three areas, since the error thus made would be an infinitesimal of higher order and would disappear in the limit process (which is, indeed, a simple process of integration). It is essential to note that this sum

$$l \cdot dp + \frac{l^2}{2} \cdot d\phi$$

agrees in sign with the area of the quadrilateral $(1, 1', 2', 2)$, if we measure $d\phi$ positively in a counterclockwise sense and dp positively for translation toward the side of increasing ϕ.

Integration along the path of motion yields for the area swept out by A_1A_2 the value

$$J = l \int dp + \frac{l^2}{2} \int d\phi\,.$$

The integral $\int d\phi$ represents the entire angle through which the rod turns with respect to its initial position. Since we returned the rod to its initial position, $\int d\phi = 0$, unless the rod has made a complete revolution, so that the area is **[13]**

$$J = l \int dp\,. \tag{1}$$

If, however, the rod made one or more complete turns before returning to its original position, which is possible with suitably chosen paths for A_1 and A_2, then $\int d\phi$ is a multiple of 2π, and we must add to the right-hand side $+\pi l^2$ for each complete turn in the positive sense and $-\pi l^2$ for each one in the negative sense. For the sake of simplicity we shall pass over this slight complication.

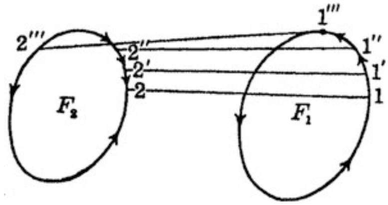

Figure 20

Now we can determine this same area J in a somewhat different way (see Fig. 20). In the succession of elementary motions let the rod take, one after another, the positions 1 2, $1'\,2'$, $1''\,2''$, ... Then J will be the sum of the elementary quadrilaterals

$$J = (1, 1', 2', 2) + (1', 1'', 2'', 2') + (1'', 1''', 2''', 2'') + \ldots,$$

or, more exactly, the integral which represents the limit of this sum, whereby each quadrilateral is to be travelled around in the sense here indicated, just as before. Using our earlier polygon formula, we now have, where *0* is the arbitrarily chosen origin of coordinates,

$$\begin{aligned} J = {} & (0, 1, 1') + (0, 1', 2') + (0, 2', 2) + (0, 2, 1) \\ & + (0, 1', 1'') + (0, 1'', 2'') + (0, 2'', 2') + (0, 2', 1') \\ & + (0, 1'', 1''') + (0, 1''', 2''') + (0, 2''', 2'') + (0, 2'', 1'') \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

The second triangle here in each row is the same as the fourth triangle in the next following row, but with opposite sense of direction,

$$[(0, 1', 2') = -(0, 2', 1'), (0, 1'', 2'') = -(0, 2'', 1''), \ldots],$$

so that these summands all cancel each other. Moreover, since the series of elementary quadrilaterals is closed, this summand $(0, 1, 2)$ will appear in the last row and will cancel $(0, 2, 1)$ of the first row. There will remain only the first and third triangles of each row. These first triangles, however, by what precedes, add up to the polygon $(1, 1', 1'', \ldots)$, and this, in the limit, is the *area F_1 of the curve described by the end A_1 of the rod*. Similarly, the third triangles, if we change the minus sign
[14] everywhere, add up to $(2, 2', 2'', \ldots)$, which, in the limit, is the *area F_2 of the curve described by A_2*. Thus we have, finally,

$$J = F_1 - F_2 . \tag{2}$$

Obviously both curves can cross over arbitrarily, provided we define F_1 and F_2 with careful regard to our sign rule.

The *geometric theory* of the planimeter is contained in the two formulas (1) and (2). If, namely, we allow A_2 to move along a curve of known area F_2 and the "marker" A_1 to glide along the boundary the curve, which encircles the searched area F_1, we can at once determine the value of

$$F_1 = F_2 + l \int dp \tag{2'}$$

if we have a *device, which allows us to measure* $\int dp$. Amsler created such a device – and that is the second, *mechanical* part of his invention – by fixing a roller upon the rod A_1A_2 as axis, which rolls upon the paper with the motion of the rod. Let its distance from A_2 be λ and its radius ρ (see Fig. 21). The angle ψ, through which the roller turns with the motion of the rod, will be the sum of the angles $d\psi$ that arise in the elementary motions. Each $d\psi$ can be thought of as made up of the rotations $d\psi_1$, $d\psi_2$, $d\psi_3$ that come from the three simple movements of the rod from which we constructed each of its elementary motions (p. [12]). During the translation (1), the roller will not turn initially, so that $d\psi_1 = 0$; during the translation (2) of A_1A_2 normal to itself, in amount dp, the roller moves over the paper in amount $dp = \rho d\psi_2$, so that $d\psi_2 = dp/\rho$; during the rotation (3) about A_2, through the angle $d\phi$, the roller rim moves in amount $\lambda\, d\phi = \rho d\psi_3$, so that $d\psi_3 = (\lambda/\rho) d\phi$. We have then, finally,

$$d\psi = \frac{1}{\rho} dp + \frac{\lambda}{\rho} d\phi .$$

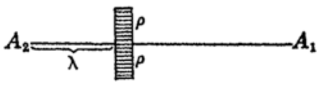

Figure 21

If we integrate over the entire path of motion, we have $\int d\phi = 0$ if A_1A_2 returns to its original position without making a complete turn, and the full turning angle of the Amsler roller will be

$$\psi = \frac{1}{\rho}\int dp\,. \tag{3}$$

If the rod, however, makes one or more complete rotations, then there will appear appropriate multiples of $2\pi(\lambda/\rho)$ on the right; but of this, again, we shall take no account.

Combining the formulas (2′) and (3), we obtain finally the formula **[15]**

$$F_1 - F_2 = l\cdot\rho\cdot\psi\,,$$

that is, the difference between the two areas encircled by the two ends of the rod is measured by the angle ψ through which the roller turns.

Figure 22

In the making of the instrument, it is desirable to *make F_2 zero*. Amsler brings this about in an admirable way by attaching A_2 to an arm, which is made to rotate about a fixed point M. (See Fig. 22.) Then A_2 can move only back and forth on the periphery of a circle and can therefore enclose no area, if we ignore the complicating possibility that A_2 makes one or more complete circuits about M. Because of this "*pole*" M, the whole instrument is often called a *polar planimeter*. The instrument is actually operated simply by causing the point A_1, provided with a *marker*, to travel around the boundary of the area one wishes to measure, and by then reading the angle ψ on the roller. We obtain thus the enclosed area $F_1 = l\cdot\rho\cdot\psi$. The constant of the instrument $l\rho$ can be determined by measuring a known area, say a unit square.

I can show you here a picture of the polar planimeter (see Fig. 23). Of course you must examine the instrument yourself, and manipulate it, if you wish fully to understand it. Naturally, if the instrument is to function reliably, it must be constructed in a manner more complicated than is implied by the theoretical discussion. In this connection, let me add a few words. The point M is carried by a heavy mass and is joined to A_2 by a rod.

The theoretically important rod A_1A_2, which we talked about, is not the second metal bar, which you see on the instrument, but the ideal prolongation of the axis of the roller, which is parallel to that bar and which passes through the moving marker point A_1. This sharp point is accompanied by a parallel blunt peg to keep

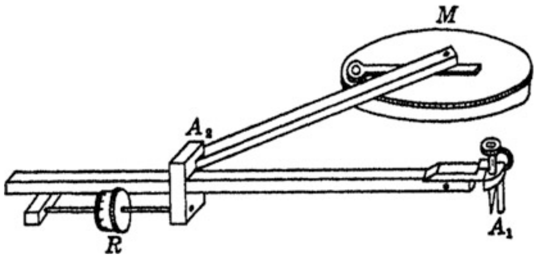

Figure 23

[16] the point A_1 from tearing the paper. The roller carries a vernier for finer readings and a marker for recording complete revolutions.

Instead of mentioning further details, I should like here to sound a general warning against neglecting, in favour of theory, the *actual putting into practice* when considering such instruments. The pure mathematician is often too prone to do so. Such neglect is just as unjustifiably one-sided as is the opposite extreme of the mechanic who, without taking an interest in the theory, loses himself in details of construction. Applied mathematics should supply here a bond of union. It should, in particular, take into account that the theoretical formulation of the principle is never exactly realised in the instrument: thus the joints of the apparatus will always be somewhat loose; the roller will always slide somewhat instead of only rolling; finally, the drawing paper is never a uniform plane, and one is never able to guide the pencil point exactly along the curve. To what extent such errors are important, to how many places, in consequence, the result read off of the roller can be relied upon, are of course questions of greatest importance in practice. To investigate such questions is the province of applied mathematics.

In connection with this digression, I shall consider the place of the present lecture course with reference to two earlier ones of a similar title, which appear likewise in lithographed form: *Applications of Differential and Integral Calculus to Geometry, a Revision of Principles* [summer term 1901; prepared by Conrad H. Müller[5]], and *Introduction to Higher Geometry* [winter term 1892–93 and summer term 1893; prepared by Friedrich Schilling[6]]. In the first one of these lecture courses, the focus is on the difference just mentioned between abstract and practical geometry. In fact we had, in the related seminar, a talk on the sources of error in Amsler's polar planimeter. In the other lecture course, however, I developed more extensively the theories of abstract geometry to meet the needs of the specialist who desires, in the spirit of research of today, to work independently in this field. In the present course, finally, I want to do a third thing: I should like to set forth, so to speak, the *elementary theoretical [das Elementartheoretische] of geometry*: those things

[5] New printing, Leipzig, 1907. [Will appear shortly as vol. III of the present edition of *Elementary Mathematics*.]
[6] Two parts. New printing, Leipzig, 1907. [Out of print. Concerning the plan for a new edition, see the preface to vol. I, p. [v].]

which, without question, every prospective teacher should know, and in particular, also, the things which are of elementary importance for applications in physics and mechanics. I shall be able to refer in this course only occasionally to things which belong to the first two fields mentioned above.

Returning now to our general considerations about areas and volumes, I shall catch up first on a *historical note*. I wish to mention the man who first applied con- **[17]** sistently the sign principle in geometry, the great geometrician August Ferdinand Möbius, of Leipzig. The book in which he took this important step is an early work, of the year 1827: *The Barycentric Calculus*.[7] It is one of the works, which are decidedly fundamental for the newer geometry. The reading of his book is unusually pleasant, if only because of the beautiful presentation. The title refers to the fact that Möbius proceeds from the following considerations, which have to do with centres of gravity. At three fixed points O_1, O_2, O_3 of a plane are placed three masses m_1, m_2, m_3 which may be positive or negative, as in the case of electric charges. Then the centre of gravity P is uniquely determined, and we can make it assume any position in the plane by varying m_1, m_2, m_3. Now the three masses m_1, m_2, m_3 are thought of as coordinates of P, so that P depends only upon the ratios of these magnitudes. This is the first instance of the introduction into geometry of what we now call *trilinear coordinates*. So much in explanation of the title of Möbius' book. As to its very interesting contents, we shall be concerned now mainly with §§ 17–20, where the principle of the sign is applied in determining the area of a triangle or the volume of a tetrahedron, and in which the definitions that I have mentioned are given.

Volumes of Polyhedral; the Law of Edges

I should remark also that Möbius, as an old man, extended these results in 1858 by a far-reaching discovery, which was first published, however, in 1865 in the paper entitled On *the determination of the volume of a polyhedron*.[8] In this he proved, namely, that *there are polyhedra to which we cannot in any way assign a volume*, whereas we can, as we saw earlier on, define a precise area for any plane polygon no matter in how complicated a manner it establishes itself. We shall now consider in detail these remarkable phenomena.

Let us start from the formula established above for the volume of the tetrahedron:

$$(1,2,3,4) = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

[7] Leipzig, 1827 – *Gesammelte Werke*, vol. I (Leipzig, 1885), 633 pages.

[8] Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften (Mathemathisch-physikalische Klasse), vol. 17 (1865), p. 31 = *Gesammelte Werke*, vol. 2 (Leipzig, 1886), p. 473.

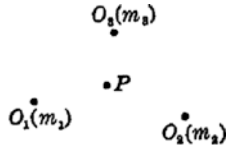

Figure 24

[18] If we expand this determinant according to minors of the last column, this amounts – as we saw earlier (p. [7]–[8]), in the case of the triangle – to resolving the tetrahedron into four others, which have the faces of the given tetrahedron as their bases and the origin as their common vertex. According to the sign rule in the theory of determinants, we shall obtain, if we take the cyclic order 1, 2, 3, 4, the following formula:

$$(1,2,3,4) = (0,2,3,4) - (0,3,4,1) + (0,4,1,2) - (0,1,2,3)\,.$$

The reason why minus signs appear, whereas, with the triangle, only plus signs occurred, is that determinants of even order change sign under cyclic interchanges, while those of odd order do not. Of course we can get rid of the minus signs by suitable interchanges of rows, but we must then give up the cyclic order. We can write, for example,

$$(1,2,3,4) = (0,2,3,4) + (0,4,3,1) + (0,4,1,2) + (0,2,1,3)\,.$$

In order to appreciate the law involved here, think of the tetrahedral faces as made of paper and as folded down into the plane $(2,3,4)$, whereby the vertex 1 takes three different positions (see Fig. 25). Then the vertices of each of the three faces appear, in the last formula, in an order, which corresponds, in Fig. 25, to a counterclockwise circuit about all the triangles.

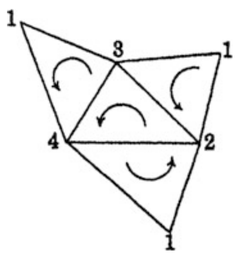

Figure 25

We can obtain the same result for this space figure, of course, without any folding down of the faces. To each of the six edges there correspond two faces, and it is clear that, *when the circuit is made about all the triangles in the order indicated*,

each side will be travelled once in one sense and once in the other. By this rule, which Möbius called the law of edges, there is obviously set up a definite sense of circuit for all the face triangles, as soon as one is arbitrarily selected for one face triangle. Our formula says now: *A tetrahedron* $(1,2,3,4)$ *can be thought of as the sum of four tetrahedra with the common first vertex* 0, *provided that after choosing the circuit sense* $(2,3,4)$ *for one triangle we select the circuit sense for the other faces according to Möbius' law of edges*.

Just as we defined the area of an arbitrary polygon earlier (p. [9]), by resolving it into triangles and generalising the triangle formula, so now we shall try to pass from the result just obtained to a *definition of the volume of an arbitrary polyhedron*. In the present case, however, we must not only allow the sides of a single polygonal face of our polyhedron to cross each other, but must also allow the faces to intersect **[19]** in an arbitrary way. We now select an arbitrary auxiliary point 0, and, as a first step, we define the *volume of one particular pyramid* which projects from 0 one of the polygonal faces of the polyhedron.

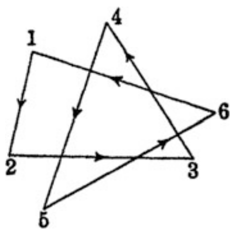

Figure 26

For this purpose we must first choose the sense of direction for its base surface. [Suppose it to be the face $(1,2,3,4,5,6)$ (Fig. 26) of the polyhedron.] This polygon has now a definite area, according to what precedes, and we shall set the volume of the pyramid equal to one-third of the product of its base by its height, as in elementary geometry, and merely add a positive or a negative sign according as the circuit $(1,2,3,4,5,6)$, viewed from 0, is travelled around counterclockwise or the reverse. We see easily that this definition includes, as a special case, the earlier agreements as to the volume of the tetrahedron. Moreover, we can deduce this definition from that special case if we replace the polygon by its component triangles, travelled around in a manner that their sum will yield its area, and then define the pyramid as the sum of the tetrahedra which these triangles project.

In order to represent the polyhedron, in the general case, as the sum of such partial pyramids, one must assign a definite direction of circuit for *each* of its faces, and the guide for this selection must be the law of edges, in view of what precedes: *We choose arbitrarily the sense of circuit for one face*, *then continue the circuits so that each edge of two contiguous faces is traversed in opposite senses*. If this process can be completed for the entire polyhedral surface without contradiction, then the *volume of the polyhedron is determined as the sum of the volumes of the partial pyramids into which the faces of the polyhedron*, *traversed in the sense*

indicated, *project from an arbitrary point* 0. It is easy to see that this determination is unique and independent of the position of 0.

One-sided Polyhedra

It is very remarkable, *however*, *that this law of edges cannot be carried out without contradiction for every closed polyhedral surface*; that is, there are polyhedra for which every attempt to fix a sign fails, and to which we cannot, therefore, assign a volume. This is the great discovery, which Möbius published in 1865. He discusses there, among others, the surface, which was later called the *Möbius band*. This surface is constructed by taking a long narrow rectangle of paper $A_1B_1A_2B_2$ (see Fig. 27) and, after a half-turn, bringing the two small ends together so that A_1
[20] coincides with A_2 and B_1 with B_2. It is clear that the front and back faces of the sheet are thus brought into connection, so that a *surface* is formed *that has only one side*. We may describe it drastically as follows:

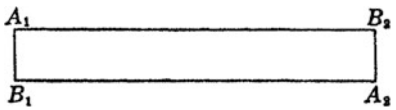

Figure 27

A painter who wished to paint the strip would find that he needed twice as much paint as he had supposed from the length of the strip; for after painting down the length of the strip, he would find himself opposite the point of beginning and he would have to go around again to reach the starting place.

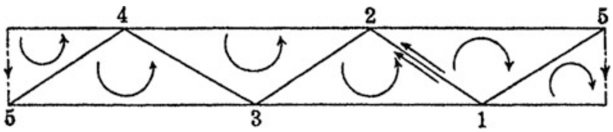

Figure 28

Instead of this curved sheet, we can set up a polyhedral surface (not closed) with plane parts of the same property, by dividing the original paper rectangle into triangles and creasing it along their edges. *To the strip of triangles thus obtained it is no longer possible to apply the law of edges*. At least five triangles are required, and they should be arranged as in Fig. 28, where the two half-triangles, right and left, form one triangle $(4, 5, 1)$ in the process of folding. If we choose here $(1, 2, 3)$ as the positive sense of circuit and continue to the left according to the law of edges, we obtain, in order, the senses $(3, 2, 4)$, $(3, 4, 5)$, $(5, 4, 1)$, $(5, 1, 2)$, so that finally 1 2 is traversed in the *same* sense as in $(1, 2, 3)$, which contradicts the law of edges.

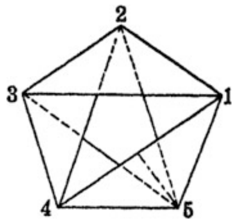

Figure 29

Looked at from above, the folded strip appears as a figure with five vertices and with the five sides 1 3, 3 5, 5 2, 2 4, 4 1 as diagonals, as sketched in the adjoining figure (Fig. 29). With this zone of triangles Möbius constructs a *closed polyhedron* by joining its free edges - these five diagonals – by means of triangles with an arbitrary point in space 0, most suitably chosen above the middle of the pentagon. In other words, he sets up a five-sided pyramid with faces crossed over It is, of course, likewise impossible to apply the law of edges to this closed polyhedron with ten triangular faces, so that we cannot talk about its *volume*.[9]

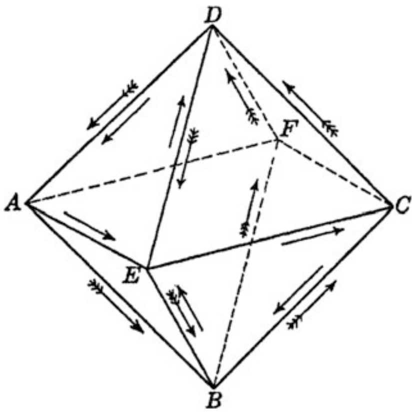

Figure 30

Another one-sided polyhedron, which is closed and simple in construction, can be obtained easily in the following way from an octahedron *ABCDEF* (see Fig. 30). **[21]** Select four faces of the octahedron that are not consecutive, that have, thus, a vertex but no edge in common (say *AED*, *EBC*, *CFD*, *ABF*), and the three diagonal planes *ABCD*, *EBFD*, *AECF*. The "heptahedron"[10] so formed has the same edges as the

[9] Compare the application in graphical statics of this one-sided polyhedron in my paper *Ueber Selbstspannungen ebener Diagramme*, Mathematische Annalen, vol. 67, p. 438 [= Klein, F., *Gesammelte Mathematische Abhandlungen*, vol. 2, p. 692, Berlin, 1922],

[10] [First mentioned in the literature by C. Reinhardt, *Zu Möbius' Polyedertheorie*, Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften (mathematisch-physikalische Klasse), vol. 37, 1885.]

octahedron, for in every edge of the latter two contiguous faces of the heptahedron meet (namely, a face and a diagonal plane of the octahedron). The diagonals of the octahedron are not to be considered as edges of this heptahedron since for it the diagonal planes are not consecutive. The diagonals *AC*, *BD*, *EF* are, rather, lines along which the heptahedron intersects itself. We can prove the one-sidedness of this heptahedron by using again the law of edges. If we pick out, namely, the successive faces *AED*, *EDFB*, *ECB*, *ABCD*, assign for the first one a sense of circuit, and determine the sense for the others by the law of edges, it turns out that the edge *AD is* traversed twice in the same sense.

With this I bring to a close the consideration of numbers as the measure of areas, and pass on to the treatment of *additional geometric elementary quantities*. Just as the name Möbius has guided us thus far, we shall now follow the thoughts of the great Stettin geometrician, Hermann Graßmann, as he first set them down in 1844 in his *Lineale Ausdehnungslehre*.[11] This book, like that of Möbius, is rich in ideas, but, unlike Möbius' book, it is written in a style that is extraordinarily obscure, so that for decades it was not considered nor understood. Only when similar trains of thought came from other sources were they recognised belatedly in Graßmann's book. If you wish to get an impression of this abstract manner of writing, you need
[22] only glance at the chapter headings of this book's introduction. They are: *Derivation of the Notion of Pure Mathematics*, *Deduction of the Theory of Extension*, *Exposition of the Theory of Extension*, *Form of Presentation* – then there follows *Survey of the General Theory of Forms*. Only after you have fought your way through these expositions, will you come to the purely abstract presentation of the subject itself, which is still very hard to understand. It was not until a later revision of the Ausdehnungslehre[12] appeared in 1862 that Graßmann used a somewhat more accessible, analytic presentation, with the use of coordinates. Moreover, Graßmann coined the word *Ausdehnungslehre* (theory of extension) to imply that his developments were applicable to any number of dimensions, while geometry was, for him, the application of this new entirely abstract discipline to the ordinary space of three dimensions. This new word did not, however, take root. One speaks today briefly of *n-dimensional geometry*.

Let us make use of our familiarity with analytic coordinates in forming an acquaintance with the Graßmann notions. Confining ourselves, first, to plane geometry, we shall use the Graßmann Principle as the title of the next chapter.

[11] Leipzig, 1844. See *Gesammelte mathematische und physikalische Werke*, vol. 1 (Leipzig, 1894), 2nd edition, Leipzig, 1898.
[12] Berlin, 1862. See *Werke*, vol. 1, Part 2, Leipzig, 1896.

II. The Graßmannian Determinant Principle for the Plane

Let us recall the fundaments of the considerations of the first chapter. There, using the coordinates of three points, we set up the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

and interpreted it as twice the area of a triangle, i.e., as the area of a parallelogram. Now let us consider, in addition, the schemata made with two points, and with one point, respectively:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} x_1 & y_1 & 1 \end{vmatrix}$$

which we call *matrices*. Every such matrix is to represent the *totality of determinants which can be made from it by omitting one column*, *or two columns*, *respectively*. Thus we obtain from the first matrix, by omitting the first and then the second column, the two-rowed determinants

$$Y = y_1 - y_2\,, \quad X = x_1 - x_2$$

and by omitting the third column, the determinant $N = x_1y_2 - x_2y_1$. The notation is chosen so that it will be appropriate for geometry of space. We must inquire what geometric configuration is determined by these three determinants X, Y, and N. We shall look upon this configuration as a new elementary geometric magnitude that has the same justification so far as the area of the triangle. From the second one-rowed matrix, we get, as one-rowed determinants, beside the number 1, the coordinates (x_1, y_1) themselves. They determine the point, which has these coordinates as the simplest elementary magnitude, and they require no further investigation. [23]

It will now be comprehensible if I give a general enunciation of the Graßmann principle: *We consider*, *in the plane*, *as well as in space*, *all matrices* (*with fewer rows than columns*) *whose rows are formed from the coordinates of a point and* 1, *and we inquire what geometric configurations are determined by the determinants which result when we omit a sufficient number of columns*.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_3

In this principle, which is here set up somewhat arbitrarily, and which only gradually will disclose itself as a useful guide through the mass of basic geometric configurations, we shall recognise eventually a natural development of an extensive group of ideas which embrace the entire geometric systematic structure.

Line-bound Vectors

But let us return to the concrete problem: What is given in the figure (see Fig. 31) of two points 1 and 2, if we know the determinants X, Y, and N? Obviously there remains still one degree of freedom in the position of the points, since it takes four magnitudes to fix them.

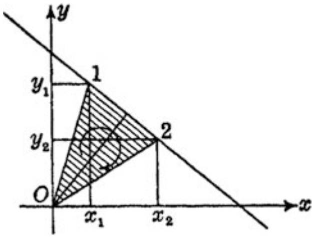

Figure 31

I assert: *We obtain the same triple of values* X, Y, *and* N *if, and only if,* 1 *is the endpoint and* 2 *the initial point of a segment, with definite length and direction, which is free to move on a definite straight line*. Here, as well as in what follows, we think of the arrow as placed so as to indicate direction from the initial point 2 toward the endpoint 1.

That the line joining 1 and 2 is determined by X, Y, and N follows at once from the fact that its equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

can be written in the form $Y \cdot x - X \cdot y + N = 0$. From this one sees also that this
[24] *line is determined if only the ratios* $X : Y : N$ *are known*. Furthermore, we see from our earlier consideration of length of segments and of area of triangles that X and Y represent the projections upon the x-axis and the y-axis of the segment $(1, 2)$ with the direction from 2 toward 1, and N represents twice the area of the triangle $(0, 1, 2)$ taken with the sense of circuit $0, 1, 2$. Obviously, then, the only changes in position of the points $1, 2$ which leave X, Y, and N unchanged are translations of the segment $(1, 2)$ along its line, with maintenance of its length and its sense. This proves my assertion. Graßmann called such a segment of definite length and direction lying upon a definite line a *Linienteil* (*directed line segment*). The word *vector* is more

usual today, in German literature, or to be more exact, *Linienflüchtiger Vector* (*line-bound vector*). We speak simply of a *vector*, or of a *free vector*, if the segment is allowed to move parallel to itself (under maintenance of length and sense) even outside of its line. *The line-bound vector, determined by the matrix*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0\,,$$

in other words, by the determinants X, Y, and N, is the first geometric elementary configuration that we consider according to the Graßmann principle.

Application in the Statics of Rigid Systems

I remark, at once, that the quantities X and Y by themselves, determine a *free vector*, since they are unaltered by the parallel translation of the segment outside of the line. Similarly, the ratios $X : Y : N$, equivalent to two quantities, determine only the *unlimited straight line*, not the length of a segment upon it. The free vector and the unlimited straight line are thus auxiliary configurations that we encounter here. The principle, which will guide us in the introduction of auxiliary configurations, will be developed later.

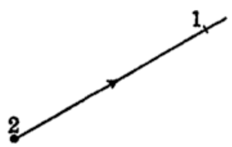

Figure 32

These notions play a very important role in mechanics in the study of *elementary statics*, where, traditionally, they have presented themselves naturally on their own account. As long as we operate in the plane, we shall be concerned here with the *statics of plane rigid systems*. For geometric treatment, one can consider the *Linienteil* (*directed line segment*) as the full equivalent of the *force,* which is applied to the system, the point of application of which may be moved at will in the direction of the force because of the rigidity of the body. Let us represent the force here in the spirit of the old mechanics: A rope is attached at the point 2 and a pull is given whose intensity is measured by the segment 1 2 (see Fig. 32). I recall, as an example of the vivid way of thinking in the old mechanics, in contrast to the abstract modern way of presentation that there always used to be the picture of a hand pulling on the rope.[13] Of the coordinates of the directed line segment (X, Y, N), *the first two* **[25]**
are called the components of the force, while N is the moment of turning about

[13] See, for example, the tables in Pierre Varignon, *Nouvelle Mécanique ou Statique*, Paris, 1775.

O. For, from the equation of the line one gets the perpendicular upon it from O as $p = N/\sqrt{x^2 + y^2}$ so that N is actually the product of the distance p and the length $\sqrt{x^2 + y^2}$ of the segment, i.e., the magnitude of the force. We can consider these three magnitudes together as the coordinates of the force. The analytic definition gives for them in every case – this is especially important – well-determined signs, which we can interpret geometrically, just as before. To be sure, it should be noted here that, in deference to the symmetry of the formulas, we have departed from the customary method in mechanics of determining the sign of the turning moment. In fact, it is customary to use the determinant formed from the coordinates of the initial point 2 and the two coordinates (X, Y) of the free vector:

$$\begin{vmatrix} x_2 & y_2 \\ X & Y \end{vmatrix} = \begin{vmatrix} x_2 & y_2 \\ x_1 - x_2 & y_1 - y_2 \end{vmatrix},$$

which obviously is equal and opposite to our N. But this small discrepancy can hardly give rise to confusion, if it is once known.

The first problem of the mechanics of rigid bodies is to find the resultant of an arbitrary system of such forces (X_i, Y_i, N_i), $(i = 1, 2, \ldots, n)$. This amounts, analytically, to forming the line-bound vector with the coordinates

$$\sum_{i=1}^{n} X_i\,, \quad \sum_{i=1}^{n} Y_i\,, \quad \sum_{i=1}^{n} N_i\,.$$

Very elegant methods for the geometric solution of this problem are established in *graphical statics*. With two forces, we use simply the well-known parallelogram law, while for $n > 2$, we have to do with the "polygon of forces" and the "equilibrium polygon". In general, we find a unique line-bound vector as the resultant of any system of forces. There are, however, exceptions, for example, where the system consists of two parallel forces which are equal and are oppositely directed on two different lines, (X, Y, N_1), and $(-X, -Y, N_2)$, $(N_1 \neq -N_2)$. The resultant has the components $(0, 0, N_1 + N_2)$, numbers, which obviously can never be the coordinates of a vector. The elementary presentation can do nothing with this phenomenon and must always expect the appearance of such irreducible, so-called *couples*, which always disturb the simplicity and generality of the theorems. We can easily fit these apparent exceptions into our system, however, if we consider that our earlier for-
[26] mulas, applied formally to the components $(0, 0, N_1 + N_2)$, yield $\sqrt{0^2 + 0^2} = 0$ as the intensity of the resultant and

$$p = \frac{N_1 + N_2}{0} = \infty$$

as its distance from the origin. Thus, if, in the case of an ordinary force, one allows its distance p from the origin to become infinite and its intensity $\sqrt{X^2 + Y^2}$ to approach zero so that the product $p \cdot \sqrt{X^2 + Y^2}$ which is the turning moment, remains finite, the components assume precisely those exceptional values, so that one can look upon the *resultant* $(0, 0, N_1 + N_2)$ *of a couple as an infinitesimal*

but infinitely remote force with a finite turning moment. This fiction is extremely convenient and useful for the advancing science, and corresponds entirely to the customary introduction of infinitely remote elements into geometry. Above all, we are able, on the basis of this extension of the notion of force, to enunciate the *perfectly general* theorem that an *arbitrary number of forces acting in a plane have*, *in all cases*, *a single force as a resultant*, whereas in the elementary presentation one must always drag along the alternative concept of a couple.

Classification of Geometric Quantities Under Transformation of the Rectangular Coordinates

Let me now complete our discussions by studying the *behaviour of our elementary quantities under transformation of the rectangular coordinates*. That will supply a valuable principle of classification for the application, in its finer shades, of the Graßmann system.

The *formulas for the change of coordinates*, i.e., the expressions for (x', y'), the coordinates of the point for the new position of the axes, in terms of the original coordinates (x, y), for the four fundamental transformations of rectangular coordinate systems are as follows:

1. For *parallel translation*:

$$(A_1) \qquad \begin{cases} x' = x + a\,, \\ y' = y + b\,. \end{cases}$$

2. For *rotation through an angle ϕ*:

$$(A_2) \qquad \begin{cases} x' = x\ \cos\phi + y\ \sin\phi\,, \\ y' = -x\ \sin\phi + y\ \cos\phi\,. \end{cases}$$

3. For *reflection in the x-axis*:

$$(A_3) \qquad x' = x\,, \quad y' = -y\,.$$

4. For *a change in the unit of measure*:

$$(A_4) \qquad x' = \lambda x\,, \quad y' = \lambda y\,.$$

If we combine with one another transformations of these four sorts for all values of the parameters a, b, ϕ, λ, we obtain the equations for the *most general transition possible from one rectangular coordinate system to another* with simultaneous change of unit. The combination of all possible translations and rotations corresponds to the totality of *proper movements* of the coordinate system within the plane. The totality of these transformations forms a *group*, i.e., the combination of any two of them gives again a transformation of the totality, and the inverse of any **[27]**

transformation is always represented. The special transformations (A) from which all the others can be derived are called *generators of the group*.

Before we inquire how these separate transformations change our determinants X, Y, and N, I shall enunciate *two general principles*, which I have habitually emphasised and have put into the foreground in these fundamental geometric discussions. Although in this generality they sound at first somewhat obscure, they will, with concrete illustrations, soon become clear. One of them is *that the geometric properties of any figure must be expressible in formulas which are not changed when one changes the coordinate system*, i.e., when one subjects all the points of the figure simultaneously to one of our transformations; *and*, *conversely*, *any formula which*, *in this sense*, *is invariant under the group of these coordinate transformations must represent a geometric property*. As simplest examples, which all of you know, let me remind you of the expression for the distance or for the angle, in the figure of two points or of two lines. We shall have to do repeatedly with these and with many other similar formulas in the following pages. For the sake of clearness, I shall give a trivial example of non-invariant formulas: The equation $y = 0$, for the figure consisting of the point (x, y) of the plane, says that this point lies on the x-axis, which is, after all, a thoroughly arbitrary addition, foreign to the nature of the figure, useful only in serving to describe it. Likewise, every non-invariant equation represents some relation of the figure to external, arbitrarily added, things, in particular to the coordinate system, but it does not represent any geometric property of the figure.

The *second principle* has to do with a system of analytic magnitudes, which are formed from the coordinates of several points $1, 2, \ldots$, such as our 3 quantities X, Y, and N, for example. If this system has the property of transforming into itself, in a definite way, under a transformation of coordinates, i.e., *if the system of magnitudes formed from the new coordinates* of the points $1, 2, \ldots$, *expresses itself in terms exclusively of these magnitudes* formed in the same way from the old coordinates (the coordinates themselves not appearing explicitly), *then we say that the system defines a new geometric configuration*, *i.e.*, *one which is independent of the coordinate system. In fact*, *we shall classify all analytic expressions according to their behaviour under coordinate transformation*, *and we shall define as geometrically equivalent two series of expressions, which transform in the same way.*

[28]

Application of the Classification Principle to the Elementary Quantities

We shall now make all this clear, using the material supplied by the Graßmann elementary magnitudes. To that end, we subject our two points (x_1, y_1), (x_2, y_2) to the same coordinate transformation.

1. Let us begin with the *translation* (A_1):

$$x_1' = x_1 + a\,, \quad x_2' = x_2 + a\,,$$
$$y_1' = y_1 + b\,, \quad y_2' = y_2 + b\,.$$

Comparing the coordinates of the vector before and after the transformation, we have

$$X = x_1 - x_2\,, \quad Y = y_1 - y_2\,, N = x_1 y_2 - x_2 y_1\,,$$
$$X' = x'_1 - x'_2\,, \quad Y' = y'_1 - y'_2\,, \quad N' = x'_1 y'_2 - x'_2 y'_1\,.$$

It follows immediately that

$$(B_1) \qquad \begin{cases} X' = X\,, \\ Y' = Y\,, \\ N' = N + bX - aY\,. \end{cases}$$

In precisely the same way, we obtain as transformation formulas:

2. Upon *rotation* (A_2):

$$(B_2) \qquad \begin{cases} X' = X\ \cos\phi + Y\ \sin\phi\,, \\ Y' = -X\ \sin\phi + Y\ \cos\phi\,, \\ N' = N\,. \end{cases}$$

3. Upon *reflection* (A_3):

$$(B_3) \qquad \begin{cases} X' = X\,, \\ Y' = -Y\,, \\ N' = -N\,. \end{cases}$$

4. Upon *change of unit of length* (A_4):

$$(B_4) \qquad \begin{cases} X' = \lambda X\,, \\ Y' = \lambda Y\,, \\ N' = \lambda^2 N\,. \end{cases}$$

In the last formulas (B_4), there is a difference in the behaviour of the magnitudes, in that the exponent of λ in the multiplying factor is not always the same. We **[29]** express this difference in physics by introducing the notion of dimension: X and Y have the dimension 1, of a line; N the dimension 2, of an area.

When we examine these four groups of formulas, we notice that the vector (directed line segment) defined by the three determinants X, Y, and N actually satisfies our definition of a geometric magnitude. The new coordinates X', Y', and N' express themselves exclusively in terms of X, Y, and N.

We see more if throughout we look at the first two equations only, into which N does not enter. The two coordinates (X', Y') of the vector in the new coordinate system depend solely upon the original values (X, Y) of these coordinates; in particular, they are unchanged under parallel translation, and, in the other cases, the relation of (X, Y) to (X', Y') is just the same as that of (x, y) to (x', y'). In view

of the second principle, enunciated above, we can say *that the two coordinates X and Y determine a geometric configuration independently of the coordinate system*, and we know already that this configuration is the *free vector*. We have thus found the formerly announced systematic principle that occasions the introduction of this configuration alongside of the vector (Linienteil).

The following consideration lies in the same field. Since X', Y', and N' occur, in all four groups of formulas, as *homogeneous* linear functions of X, Y, and N, we see, by division of each two equations, that the ratios $X'\colon Y'\colon N'$ depend also only on the ratios $X\colon Y\colon N$. Thus *these ratios* $X\colon Y\colon N$ *determine a geometric configuration independently of the coordinate system*, without regard to the actual values of the three quantities themselves, and we recognised this configuration earlier on as the *unlimited straight line*.

Let us now apply our formulas (B), in particular, to a *couple*, for which

$$X = 0\,, \quad Y = 0\,.$$

Then, of course,

$$X' = 0\,, \quad Y' = 0\,,$$

while in the four separate cases:

$$(C_1) \qquad N' = N,$$

$$(C_2) \qquad N' = N,$$

$$(C_3) \qquad N' = -N,$$

$$(C_4) \qquad N' = \lambda^2 N\,.$$

If we use the customary expression *invariant* for a quantity, which changes, under the operations of a group of transformations, at most by a factor, and if we call the invariant *absolute* or *relative* according as this factor is 1 or not, we can express
[30] formulas (C) in these words: The *turning moment of a couple is a relative invariant with respect to all rectangular coordinate transformations in the plane*.

Let us compare with this the behaviour under coordinate transformation of the elementary geometric quantity, which we studied at the beginning, the *area of the triangle*:

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\,.$$

The translation (A_1) does not change this determinant, since it only adds a to the elements of the first column and b to those of the second, i.e., the a-tuple and b-tuple, respectively, of the elements of the third column. Consequently we have

$$(D_1) \qquad \Delta' = \Delta\,.$$

Similarly, the three other transformations yield

$$(D_2) \qquad \Delta' = \Delta,$$
$$(D_3) \qquad \Delta' = -\Delta,$$
$$(D_4) \qquad \Delta' = \lambda^2 \Delta\,,$$

all of which we might easily infer at once from the geometric significance of the area of the triangle. However, these formulas agree precisely with (C): *The area of a triangle and hence every area* (which can always, indeed, be expressed as the sum of triangles) *behaves under arbitrary transformation of coordinates precisely as does the turning moment of a couple*. According to our second general principle, we may look upon both things, therefore, as equivalent geometrically, and we can interpret this statement in the following way: If we have in the plane any couple with turning moment N, and if we define, in any way, a triangle with area $\Delta = N$, this equality is preserved under all coordinate transformations, i.e., *we can illustrate the turning moment of a couple*, *regardless of the system of coordinates*, *by the area of a triangle*, *or by the area of a parallelogram*, *or by the area of any other plane figure*. Just how this geometric correspondence is to be brought about, will appear later when we come to the analogous, but somewhat more complicated, and therefore more instructive, relations in space.

With this I shall leave the geometry of the plane, in which these abstractions are almost trivially simple. To every analytic formula one can assign a good geometric meaning, whereby full analytic generality finds its way automatically into geometry. In this connection, an essential assumption, which must again be emphasised, is that the proper conventions should be made concerning the signs of the geometric configurations.

III. The Graßmannian Principle for Space [31]

"Linienteil" and "Ebenenteil"

We shall now carry out the corresponding investigations for space in complete analogy with the foregoing considerations for the plane. We start therefore from the matrices, which can be formed with the coordinates of 1, 2, 3, or 4 points:

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}, \quad \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The determinants of the first matrix represent the point coordinates themselves and require no further consideration. The fourth matrix is already a four-rowed determinant, and gives, as we know, the six-fold volume of the tetrahedron $(1, 2, 3, 4)$, which we can call a *space segment* (*Raumteil*) in agreement with the terminology to be introduced later. We can, moreover, think of it simply as the volume of a parallelepipedon with the edges 4 1, 4 2, 4 3 (see Fig. 33), which Graßmann called a *Spat* (the word *Spat* is taken from the miners' word *Kalkspat*).

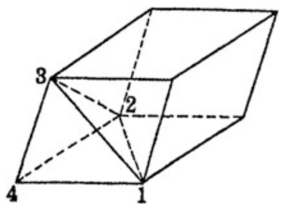

Figure 33

New configurations are supplied by the second matrix and by the third matrix. The two-rowed matrix represents the aggregate of the each following *six determinants of second order*, which arise by the deletion of two columns:

$$(1) \qquad \begin{cases} X = x_1 - x_2\,, & Y = y_1 - y_2\,, & Z = z_1 - z_2\,, \\ L = y_1 z_2 - y_2 z_1\,, & M = z_1 x_2 - z_2 x_1\,, & N = x_1 y_2 - x_2 y_1\,. \end{cases}$$

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_4

similarly, the third matrix represents the following *four determinants of third order*:

$$
(2)\qquad \begin{cases} \mathcal{L} = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, & \mathcal{M} = \begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix}, \\ \mathcal{N} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, & \mathcal{P} = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \end{cases}
$$

First, as to the six determinants (1), we can infer, from the corresponding discussion for the plane, that X, Y, and Z are the projections upon the coordinate axes of the segment joining 2 to 1, while L, M, and N are double the areas of the projec-
[32] tions upon the coordinate planes of the triangle $(0, 1, 2)$, taken in the sense $0, 1, 2$ (see Fig. 34). All these quantities remain obviously unchanged when we move the segment $(1, 2)$ along its line, preserving its length and its direction. They represent what we shall call a *directed line segment* (*Linienteil*) or *line-bound vector* (*linienflüchtiger Vector*) *of space*. The quantities X, Y, and Z themselves remain unchanged if one moves the vector out of its line parallel to itself; they therefore determine a *free vector*. Similarly the five ratios $X : Y : Z : L : M : N$ are not changed by arbitrarily changing the length or direction of the directed line segment on its line. Thus they determine the *unlimited straight line*.

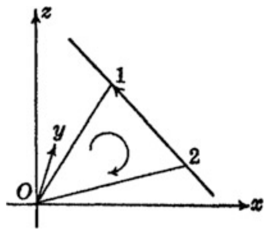

Figure 34

The four determinants (2) determine, first of all, the plane of the three points $1, 2, 3$; for we can write the equation

$$
\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0
$$

obviously in the form

$$
\mathcal{L}x + \mathcal{M}y + \mathcal{N}z + \mathcal{P} = 0 .
$$

Hence the ratios $\mathcal{L} : \mathcal{M} : \mathcal{N} : \mathcal{P}$ determine the *unlimited plane*. We see, further, that $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are double the areas of the projections upon the coordinate planes of

the triangle $(1,2,3)$, always taken in the direction $1,2,3$, and that $\mathcal{P}$ is six times the volume of the tetrahedron $(0,1,2,3)$, again with that sign which corresponds to this succession of vertices. Now these four quantities obviously are unchanged when, and only when, the triangle $(1,2,3)$ is so moved and deformed in its plane that its area and its direction is unchanged, and they determine thus a triangle or a plane area with this freedom of motion, which Graßmann calls a *plane segment* (*Ebenenteil*) or a *plane quantity* (*Plangröße*). The first three coordinates $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ of the plane segment also remain unchanged when we move the plane of the triangle parallel to itself. They determine then, as to area and direction, a triangle, which is free to move in space parallel to itself, a so-called *free plane quantity*.

If we turn now to a closer examination of the *directed line segment* we notice first that it is determined in space by five variable parameters, since its two endpoints have together six coordinates, but the one endpoint can be moved arbitrarily along a straight line. Thus the six coordinates X, Y, Z, L, M, and N of the *directed line segment*, which we defined above, cannot be independent of one another, but must satisfy a condition. We can deduce this condition most simply from the laws **[33]**
of determinants, which are, indeed, always the key to our theories. We consider the determinant which vanishes identically because two rows coincide, element for element.

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0,$$

We expand it as the sum of products of corresponding minors of the first and last pairs of rows. The first summand, which contains the two enclosed minors, is simply $N \cdot Z$, and for the whole determinant we get $2(N \cdot Z + M \cdot Y + L \cdot X)$. Hence we have the identity

$$X \cdot L + Y \cdot M + Z \cdot N = 0 \tag{3}$$

as the necessary condition for the six coordinates of any directed line segment. It is easy to show that the equation (3), between the six quantities, suffices in order for them to represent, by means of formulas (1), the coordinates of a directed line segment. I hardly need to go into this very elementary discussion.

Application to Statics of Rigid Bodies

I shall now go over again to the *application of these notions to mechanics*. Just as in the plane (p. [24]), we now have the directed line segment representing a *force applied to a rigid body in space*, including the point of application, the quantity, and the direction. Of the six coordinates of the directed line segment, we call X,

Y, and Z *the components of the force parallel to the coordinate axes* and L, M, and N *the turning moments about these axes*.[14] The three components X, Y, and Z determine the magnitude and direction of the force, whose direction-cosines are in the ratios $X : Y : Z$. We obtain the force as the diagonal of the parallelepiped whose edges are the segments X, Y, and Z on the coordinate axes. With the same construction, using L, M, and N, we get a definite direction called the *direction of the axes of the resultant turning moment*. The equation of condition (3) shows, according to a well-known formula of space geometry, *that the direction of the force and that of the axis of the resultant turning moment are at right angles to each other*. Just as in the plane, so here we shall include, as *couple*, the limiting
[34] case where $X = Y = Z = 0$, while L, M, and N do not all vanish, into the notion of directed line segment. A simple passage to the limit shows that one should mean here an *infinitely remote infinitesimal force whose turning moments remain finite*. The elementary theory avoids this form of expression and looks upon a couple only as the combination of two equal, oppositely directed, forces acting upon different parallel lines: (X, Y, Z, L_1, M_1, N_1) and $(-X, -Y, -Z, L_2, M_2, N_2)$, whose sum gives, in fact, just such coordinates $(0, 0, 0, L_1 + L_2, M_1 + M_2, N_1 + N_2)$, as we have just assumed.

We have to consider now the *composition of a system of arbitrary forces acting upon a rigid body*: $X_i, Y_i, Z_i, L_i, M_i, N_i$, $(i = 1, 2, \ldots, n)$. Much time is spent on this problem in elementary books and lecture courses, whereas we can dispose of it rapidly here because our analytic formulas make superfluous that consideration of separate cases which the neglect of the rule of signs imposes upon the tedious elementary discussion. The fundamental principle of composition is that we set up the sums:

$$\Xi = \sum_{i=1}^{n} X_i\,, \quad \mathsf{H} = \sum_{i=1}^{n} Y_i\,, \quad \mathsf{Z} = \sum_{i=1}^{n} Z_i\,,$$
$$\Lambda = \sum_{i=1}^{n} L_i\,, \quad \mathsf{M} = \sum_{i=1}^{n} M_i\,, \quad \mathsf{N} = \sum_{i=1}^{n} N_i\,,$$

and consider them as the *coordinates of the system of forces* or, according to an appropriate term introduced by *Julius Plücker*, as *coordinates of the dyname*.[15] Here, again, we distinguish the three *components along the axes* and the three *turning moments about them*. Now this system of forces will not, in general, be a *single force*, since the six sums will not necessarily satisfy the condition for the coordinates of a single directed line segment:

$$\Xi \cdot \Lambda + \mathsf{H} \cdot \mathsf{M} + \mathsf{Z} \cdot \mathsf{N} = 0\,.$$

This is the new thing that comes up in space as opposed to the plane, namely, that a *system of forces acting upon a rigid body does not necessarily reduce to* o n e *single force*.

[14] Again we have chosen the sign opposite to that which is usually taken in mechanics.
[15] [Translator's note: dynamis is a Greek term, meaning force.]

In order to gain a concrete picture of the nature of a system of forces, we shall try to represent it in the simplest possible way as the resultant of the fewest possible forces. We shall prove *that we can consider every system as the resultant of a single force and of a couple whose axis is parallel to the line of action of that force*, *the so-called central axis of the system*; *and this resolution is unique*. This theory of the composition of forces acting upon rigid bodies had achieved its classical presentation in Louis Poinsot's *Elements de statique*, which appeared first in 1804, **[35]** and which, since then, has gone through new editions.[16] We speak, indeed, of *Poinsot's central axis*. The treatment by Poinsot was an elementary geometric one, and was very involved, just as it still is in elementary instruction.

To *prove*, now, the above theorem, we note that any single force which could arise by the withdrawal of a couple from the system must have $\mathsf{\Xi}$, H, and Z as components parallel to the axes. Thus the turning moments of the couple must be proportional to $\mathsf{\Xi}$, H, and Z if its axis is to be parallel to the central axis. We assume its six coordinates to be $0, 0, 0, k\mathsf{\Xi}$, $k\mathsf{H}$, $k\mathsf{Z}$, where k is a parameter still to be determined. To get from this couple our *dyname* $(\mathsf{\Xi}, \mathsf{H}, \mathsf{Z}, \mathsf{\Lambda}, \mathsf{M}, \mathsf{N})$, we must add to it the *dyname*

$$\mathsf{\Xi}, \mathsf{H}, \mathsf{Z}, \mathsf{\Lambda} - k\mathsf{\Xi}, \mathsf{M} - k\mathsf{H}, \mathsf{N} - k\mathsf{Z}\,.$$

The theorem would be proved if one could determine k so that this system would be a single force. A necessary and sufficient condition for this is that the coordinates satisfy (3), i.e., that

$$\mathsf{\Xi}(\mathsf{\Lambda} - k\mathsf{\Xi}) + \mathsf{H}(\mathsf{M} - k\mathsf{H}) + \mathsf{Z}(\mathsf{N} - k\mathsf{Z}) = 0\,.$$

From this we get uniquely

$$k = \frac{\mathsf{\Xi}\mathsf{\Lambda} + \mathsf{H}\mathsf{M} + \mathsf{Z}\mathsf{N}}{\mathsf{\Xi}^2 + \mathsf{H}^2 + \mathsf{Z}^2}\,,$$

for we may assume that the denominator is different from zero, otherwise we should be dealing with a couple instead of with a proper *dyname*. If one assigns to k this value, which Plücker calls the *parameter of the dyname*, one actually resolves the system into a couple and a single force, and the method of proof shows that the resolution is unique.

Relations to the Null-System of Möbius

Now the question arises as to what *geometric representations* one can associate with this resolution. These investigations go back again to Möbius, to his *Lehrbuch der Statik*[17] of 1837. Here he inquires about an *axis around which the turning moment of the system would be zero*, the so-called *null-axis*. The system of all these null-axes he calls a *null-system*. It is in this connection that this word, no doubt familiar to you, has its origin.

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[16] Twelfth edition by Jules Bertrand, Paris, 1877.

[17] Leipzig, 1837. See *Werke*, vol. 3, Leipzig, 1896.

[36] We must now define the *general notion of turning moment*, or *moment*, which finds application here. Let two directed line segments $(1, 2)$ and $(1', 2')$ be given in space (see Fig. 35). Construct with them the tetrahedron $(1, 2, 1', 2')$, whose volume is

$$\frac{1}{6} \cdot \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_1' & y_1' & z_1' & 1 \\ x_2' & y_2' & z_2' & 1 \end{vmatrix} .$$

Calculating this determinant as the sum of products of minors of the first and last pairs of rows, as we did with the identically vanishing determinant (p. [33]), we get $\frac{1}{6}(XL' + YM' + ZN' + LX' + MY' + NZ')$, where $X', \ldots, N'$ are the coordinates of the directed line segment $(1', 2')$. The bilinear combination of the coordinates of both directed line segments which appear here,

$$XL' + YM' + ZN' + LX' + MY' + NZ' ,$$

will be called the *moment of one directed line segment with respect to the other*. It is equal to *six times the volume of the tetrahedron whose vertices are the endpoints of the directed line segments*, and it is consequently a geometric quantity independent of the coordinate system. If r and r' are the lengths of the directed line segments, ϕ the angle between them, and p the common perpendicular (i.e., shortest distance) to their two lines, it follows from elementary geometry that the moment is $r \cdot r' \cdot p \cdot \sin \phi$, if the sign of ϕ is properly chosen.

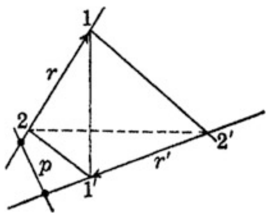

Figure 35

If, instead of the directed line segment $(1, 2)$ we choose the *unlimited straight line*, then the moment of the directed line segment $(1', 2')$ with reference to the line will be defined as its moment, in the preceding sense, taken with reference to a directed line segment of length $r = 1$ on that line, i.e., $r'\ p\ \sin \phi$. This is the result of dividing the preceding expression by $r = \left|\sqrt{X^2 + Y^2 + Z^2}\right|$ so that finally, the *moment of a directed line segment* (X', Y', Z', L', M', N') *with respect to an unlimited line, which contains the directed line segment* (X, Y, Z, L, M, N) is

$$\frac{XL' + YM' + ZN' + LX' + MY' + NZ'}{\left|\sqrt{X^2 + Y^2 + Z^2}\right|} .$$

This value depends, in fact, only upon the ratios of the six quantities $X, \ldots, N$, along with a sign common to them, so that it is fully determined when the *unlimited*

straight line and a direction on it are known. This moment is precisely what is known in statics as the *turning moment of a force*, *represented by a directed line segment*, *about the line as axis*, although a different sign is commonly chosen (see p. [33]). **[37]**

We shall now consider the *moment*, *or turning moment*, *of a system of forces*, *of a dyname*,

$$\Xi = \sum_{i=1}^{n} X_i', \ldots, \mathsf{N} = \sum_{i=1}^{n} N_i' \,.$$

By this we shall naturally mean the sum of the moments of the several forces, i.e., the expression

$$\begin{cases} \sum_{i=1}^{n} \dfrac{XL_i'+YM_i'+ZN_i'+LX_i'+MY_i'+NZ_i'}{\left|\sqrt{X^2+Y^2+Z^2}\right|} \\ \qquad = \dfrac{X\Lambda+Y\mathsf{M}+Z\mathsf{N}+L\Xi+M\mathsf{H}+N\mathsf{Z}}{\left|\sqrt{X^2+Y^2+Z^2}\right|} \,. \end{cases}$$

If, in this expression, we identify the unlimited straight line of $X, \ldots, N$ with the three positive axes, in order, the expression takes on, in order, the values $\Lambda, \mathsf{M}, \mathsf{N}$, which justifies the designations for these quantities, which we used previously (p. [34]).

Now we can take up the question raised by Möbius. A given system $\Xi, \mathsf{H}, \ldots, \mathsf{N}$ has the moment 0 with respect to a straight line $(X : Y : \ldots : N)$ (this is the *null-axis*) if

$$\Lambda X + \mathsf{M}Y + \mathsf{N}Z + \Xi L + \mathsf{H}M + \mathsf{Z}N = 0 \,.$$

Thus the *null-system of the dyname is the totality of the straight lines* $(X : Y : \ldots : N)$ *given by this equation*. But that is the most general linear homogeneous equation for the six quantities $X, \ldots, N$, since the coefficients $\Lambda, \ldots, \mathsf{Z}$, as coordinates of a dyname, can be six arbitrary quantities. Now Plücker, along with Möbius, the pioneer in analytic geometry of the nineteenth century, investigated just such totalities of straight lines, which are defined by an arbitrary linear homogeneous equation, in a connection which we shall discuss more fully later, and called them *linear complexes*. *Thus the null-system of Möbius is exactly the same as the line complex of Plücker*.

Geometrical Visualisation of the Null-System

We shall now try to give as *clear a picture as possible* of this null-system, although, of course, we cannot speak of a geometric figure in the proper meaning of that word, since the null-lines cover the entire space infinitely often. Nevertheless, its grouping can be understood quite simply. To this end, according to the plan always to be followed in this lecture course, we shall select the coordinate axes as conveniently as possible, which we accomplish here by choosing the *central axis of the dyname* **[38]**
as the z-axis. Since, as we know, the dyname is the resultant of a single force acting

along the central axis, and a couple with its axis parallel to that central axis, the four coordinates $\Xi, \mathsf{H}, \Lambda, \mathsf{M}$ must all vanish, by our choice of the z-axis, so that Z represents the quantity of the single force and N the turning moment of the couple about its axis. The parameter of the dyname is, therefore,

$$k = \frac{\Xi\Lambda + \mathsf{HM} + \mathsf{ZN}}{\Xi^2 + \mathsf{H}^2 + \mathsf{Z}^2} = \frac{\mathsf{N}}{\mathsf{Z}}\,.$$

The equation of the linear complex in the new coordinate system has then the simple form $\mathsf{N}Z + \mathsf{Z}N = 0$, or, after division by Z,

(1) $$k \cdot Z + N = 0\,.$$

We use this form as the basis of the rest of our discussion. If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points on a line $(X : Y : Z : L : M : N)$ of the null-system, then since $Z = z_1 - z_2$ and $N = x_1y_2 - x_2y_1$, the equation (1) gives, for the coordinates of any two points of a null-line, the condition

(2) $$k(z_1 - z_2) + (x_1y_2 - x_2y_1) = 0\,.$$

If now we keep P_2 fixed, then (2) is the equation for the coordinates (x_1, y_1, z_1) of all points P_1, which lie with P_2 on a straight line of the null-system. If, for the sake of clearness, we write, as variable coordinates, (x, y, z) in place of (x_1, y_1, z_1), we see that all the points P_1 fill a plane whose equation is

(2′) $$y_2x - x_2y + k \cdot z = kz_2\,.$$

This plane passes through the point P_2 itself, since the equation is satisfied by $x = x_2$, $y = y_2$, $z = z_2$. We have thus proved *that through any point P_2 in space there pass infinitely many null-lines, which form a family of rays* in the *plane, that fill the plane* (2′). Our task will be solved if we obtained a clear picture of the position of this plane (null-plane), which corresponds to every point P_2.

The two expressions $N = x_1y_2 - x_2y_1$, $Z = z_1 - z_2$, which occur in (2), have the property of remaining unchanged under translations of space parallel to the z-axis, as well as rotations about it; for translations leave x and y, hence also N, and likewise the difference $z_1 - z_2$, all unchanged, whereas rotations have no effect upon the z-coordinates, i.e., upon Z, and leave N, as area in the x-y-plane, unchanged.
[39] Consequently, equation (2), and therefore the *null-system* which it determines, *goes into itself under screw motions of space about the central axis* – for that is the meaning of the z-axis – *and translations along it.*

This theorem makes our task considerably easier. *If we only know which plane in the null-system belongs to any point of the positive half of the x-axis*, *then we know automatically also the null-plane which belongs to each point of space*; for, by translating that half-axis along, and turning it about the z-axis, we can bring one of its points into coincidence with any point in space, whereby, according to our theorem, the corresponding null-planes go into themselves.

In other words: *The null-planes of the points of a half-ray, which is perpendicular to this central axis have a position with reference to the ray and the central axis, which is independent of the choice of the ray.*

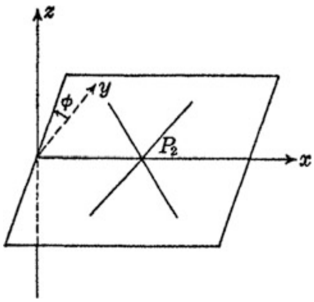

Figure 36

If we now confine ourselves to the x-axis, setting $y_2 = z_2 = 0$, we get from (2′) as the equation of the plane belonging to the point P_2 with abscissa x_2:

$$kz - x_2 y = 0\,.$$

It passes through the x-axis itself, since $y = z = 0$ satisfies the equation identically (see Fig. 36). If we write the equation in the form $z/y = x_2/k$, we infer that the angle of inclination ϕ of the plane to the horizontal (x-y-plane) has the trigonometric tangent

$$\tan\phi = \frac{x_2}{k}$$

and the position of our plane is fully determined. In Fig. 37, its trace in the vertical y-z-plane is sketched.

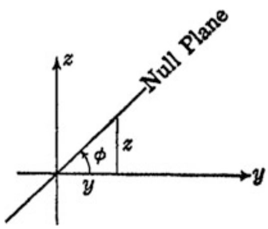

Figure 37

From what has been said above, we can state the result independently of the special choice of coordinate system: *To every point at a distance r from the central axis, thought of as vertical, there belongs a plane of the null-system, which contains the perpendicular from the point upon the axis, and whose angle of inclination to the horizontal plane has the trigonometric tangent r/k.* If we move the point on

[40] a half-ray perpendicular to the axis, then the corresponding plane of the null-system will be horizontal for $r = 0$, and will turn, with increasing r, up or down (according as $k \gtrless 0$ and will approach the vertical asymptotically when r becomes infinite. I can make these relations clearer to you by means of a *Schilling*[18] *model* (see Fig. 38) in which there is a movable arm which slides along and turns about the central axis, and which carries a plane sheet that rises in the proper way as it recedes from the axis.

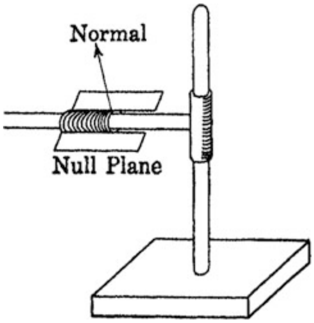

Figure 38

Let us now consider, in particular, the *direction of the normal*, which belongs to the plane through the point P_2. Its direction cosines have the same ratios as the coefficients in the equation of the plane (2′), i.e., $y_2 : (-x_2) : k$.

Connection to the Theory of Screw Movements

We can think of this same direction as the direction of motion of the point P_2 under some *infinitesimal screw movement* of space. Indeed, if we turn space as a rigid body around the z-axis through the finite angle ω and move it, at the same time, parallel to the z-axis by the amount c, every point (x, y, z) will be transferred into the new position given by the equations

$$\begin{aligned} x' &= x \cos\omega - y \sin\omega\,, \\ y' &= x \sin\omega + y \cos\omega\,, \\ z' &= z + c\,. \end{aligned}$$

We pass from this finite screw motion to an infinitesimal one by replacing ω by the infinitesimal quantity $-d\omega$ and assuming at the same time $c = k\,d\omega$. The minus sign means that for $k > 0$ the rotation in the x-y-plane is negative, if the

[18] [Translator's note: Schilling was a firm in Leipzig, which produced mathematical models; see vol. I, p. [103]]

translation is in the positive z-direction, i.e., that the screw motion is negative (left-handed). Neglecting quantities of second and higher orders in $d\omega$, that is, we have $\cos d\omega = 1$, $\sin d\omega = d\omega$, and obtain therefore:

$$x' = x + y\,d\omega\,, \quad y_1 = -x\,d\omega + y\,, \quad z' = z + k\,d\omega\,.$$

The increments of the coordinates of a definite point P_2 under this infinitesimal screw motion are $dx_2 = y_2 d\omega$, $dy_2 = -x_2 d\omega$, $dz_2 = k d\omega$, that is, P_2 will be moved **[41]** in the direction

$$dx_2 : dy_2 : dz_2 = y_2 : (-x_2) : k\,.$$

This is, in fact, precisely the direction along the normal (3). *Thus, if we give to space an infinitesimal screw motion about the central axis such that the translation along this axis is k-times the angle of rotation* (*taken negatively*), *then the plane of the null-system of parameter k which belongs to any point of space will be normal to the arc traversed by the point.*

Since the representation of a screw motion is very intuitive, we can get in this way a vivid picture of the arrangement of the planes in a null-system. For example, the greater the distance r of a point from the central axis, the longer is the horizontal projection $rd\omega$ of the element of the arc, which it traverses in the screw motion, the flatter is the path itself, since the rise, $kd\omega$, is constant, hence the steeper is the plane of the null-system, being normal to element of the arc. If we combine infinitely many of these infinitesimal screw motions into a continuous screw motion of space, then every point at a distance r from the central axis will describe a *helix* whose inclination to the horizontal has $-k/r$ for its trigonometric tangent, and whose *pitch* is therefore $2\pi k$, independent of r. *The planes normal to this helix are the planes of the null-system.*

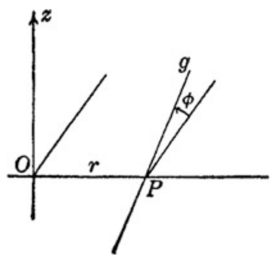

Figure 39

In conclusion, having talked only about the planes of the null-system, let us now try to get an immediately *intuitive picture of the null-axes*. We take any null-axis g (see Fig. 39) and draw its shortest distance to the central axis, i.e. the common perpendicular between g and the central axis, meeting the latter in O, and g in P. Then PO, as a perpendicular from P to the central axis, belongs to the null-system, and OPg must be the plane of the null-system belonging to P. Since g

is perpendicular to OP, it makes with the horizontal the same angle ϕ as the null-plane, i.e., $\tan\phi = r/k$, where $r = OP$. *Thus we obtain all the null-axes, if, through every point P of every half-ray perpendicular to the central axis we draw that normal to this ray, which makes with the horizontal an angle whose trigonometric tangent is* $\tan\phi = r/k$, *where r is the distance of P from the central axis.*

[42] We can make this construction still more intuitive. *We take a circular cylinder of radius r whose axis is the central axis and draw on it all helices* (see Fig. 40) *whose inclination ϕ to the horizontal plane is given by* $\tan\phi = r/k$. *The totality of tangents to these helices is obviously identical with the totality of null-axes at the distance r from the central axis. By varying r, we get all the null-axes.* As we move outward, these helices get steeper. They have at each point the corresponding null-plane as osculating plane and they are therefore at right angles to the previously mentioned helices, which are at every point normal to the null-plane.

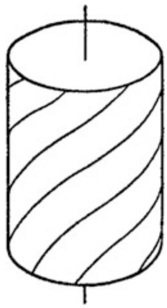

Figure 40

After this discussion, which has exhibited a double connection between helices and the null-system, we can understand why this whole theory has been associated with that of helices. Sir Robert Ball used this designation in his *Theory of Screws*,[19] in which he discussed all the geometric relations connected with a system of forces acting upon a rigid body.

Let us now return to our systematic development. We had obtained, by using Graßmann's principle, the four geometric elementary configurations, the *point*, the *line segment* (*Linienteil*), the *plane segment* (*Ebenenteil*), and the *space segment* (*Raumteil*). As in the plane, we shall now examine the behaviour of these configurations, under transformation of the rectangular coordinate system, and classify them according to the general principle announced above.

[19] Dublin, 1876.

IV. Classification of the Elementary Configurations of Space According to Their Behaviour Under Transformation of Rectangular Coordinates

General Remarks About Transformations of Rectangular Coordinate Systems in Space

Above all, of course, we should obtain a view of all possible transformations of a rectangular coordinate system in space. These transformations are really fundamental for all geometry of space, so that, for this very reason, we could not overlook them in this lecture course. The most general change in the coordinate system that comes up for consideration is made up, as in the plane, of the following component parts: (1) translation; (2) rotation about the origin; (3) reflection; (4) change in the unit of length.

The equations of *parallel translation* are, of course, **[43]**

$$(A_1) \qquad \begin{cases} x' = x + a\,, \\ y' = y + b\,, \\ z' = z + c\,. \end{cases}$$

The equations of *rotation*, in any case, have the form

$$(A_2) \qquad \begin{cases} x' = a_1 x + b_1 y + c_1 z\,, \\ y' = a_2 x + b_2 y + c_2 z\,, \\ z' = a_3 x + b_3 y + c_3 z\,. \end{cases}$$

We shall consider at once the determination of the coefficients, which is more complicated here than in the plane. The combination of all possible transformations of these two sorts yields all the *proper movements* of the coordinate system in space.

Just as, in the plane, we reflected in an axis, so here we can consider *reflection in a coordinate plane*, say the x-y-plane, and we obtain

$$x' = x\,, \quad y' = y\,, \quad z' = -z\,.$$

But we can write these formulas more symmetrically by using three minus signs, in the form

$$(A_3) \qquad x' = -x\,, \quad y' = -y\,, \quad z' = -z\,.$$

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_5

This is a *reflection in the origin*, sometimes called *inversion*.[20] In the plane,

$$x' = -x\,, \quad y' = -y$$

is not a reflection, but a rotation about 180°; and, generally, inversion in the origin is a reflection only in spaces of an odd number of dimensions. If the number is even, it is a rotation.

A *change in the unit of length*, finally, is given by the equations

$$(A_4) \qquad x' = \lambda x\,, \quad y' = \lambda y\,, \quad z' = \lambda z \quad \text{where} \quad \lambda > 0\,.$$

If $\lambda < 0$, this transformation involves a reflection, in addition to a change in unit length.

It remains for us to consider in greater detail the *formulas for rotation*. The most general rotation about the origin depends, as you know, upon three parameters, because, first, the direction cosines of the axis of rotation represent two independent quantities and, in addition, the angle of rotation is arbitrary. A symmetrical treatment of all rotations in terms of three independent parameters is furnished by the *theory of quaternions*, which you will find discussed in my lecture course[21] of last
[44] winter. Moreover, Euler had set up the formulas in question before quaternions were invented. I shall give here the treatment that one usually finds in textbooks on mechanics and which makes use of the nine direction cosines of the new axes with reference to the old. We start from the form of the equations of transformation given above:

$$(1) \qquad \begin{cases} x' = a_1 x + b_1 y + c_1 z\,, \\ y' = a_2 x + b_2 y + c_2 z\,, \\ z' = a_3 x + b_3 y + c_3 z\,. \end{cases}$$

Let us consider one point $x, y = 0, z = 0$ of the old x-axis. It has, with reference to the new system, the coordinates $x' = a_1 x$, $y' = a_2 x$, $z' = a_3 x$, that is, a_1, a_2, a_3 *are the cosines of the angles, which the new axes make with the old x-axis*. Similarly, b_1, b_2, b_3 and c_1, c_2, c_3 are the cosines of the angles, which the new axes make with the old y-axis and the old z-axis, respectively.

These nine coefficients of the equations of transformation are not at all independent of one another. We can deduce the relations between them from the interpretation just given, or we can make use of the known relations that one obtains in every *orthogonal substitution*, i.e., in every rotation or reflection with fixed origin:

$$(2) \qquad x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2\,,$$

which states that the *distance* from O is *invariant*. We shall choose the second method:

[20] Sometimes the designation "inversion" is used also for the totally different transformation by means of reciprocal radii.
[21] See Volume I, p. [64] sqq.

α) We substitute (1) in (2) and obtain, by comparing coefficients, the following six relations among the nine quantities $a_1, \ldots, c_3$:

$$\tag{3}
\begin{cases}
a_1^2 + a_2^2 + a_3^2 = 1, & b_1^2 + b_2^2 + b_3^2 = 1, & c_1^2 + c_2^2 + c_3^2 = 1, \\
b_1c_1 + b_2c_2 + b_3c_3 = 0, & c_1a_1 + c_2a_2 + c_3a_3 = 0, & a_1b_1 + a_2b_2 + a_3b_3 = 0\,.
\end{cases}$$

β) We multiply the three equations (1) by the three quantities a, b, c respectively, and add. Solving them by means of (3), we obtain

$$\tag{4}
\begin{cases}
x = a_1x' + b_1y' + c_1z'\,, \\
y = a_2x' + b_2y' + c_2z'\,, \\
z = a_3x' + b_3y' + c_3z'\,.
\end{cases}$$

This is obviously the so-called *transposed linear substitution*, which arises from (1) by interchanging rows and columns in the array of coefficients.

γ) On the other hand, solving equations (1) by the rules of determinants, we find

$$x = \frac{1}{\Delta}\begin{vmatrix} x' & b_1 & c_1 \\ y' & b_2 & c_2 \\ z' & b_3 & c_3 \end{vmatrix}, \ldots, \text{where } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The coefficient of x' here must be the same as in the first equation (4), that is, **[45]**

$$\tag{5}
\frac{1}{\Delta}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = a_1\,,$$

and similarly, *each coefficient of the orthogonal substitution must be equal to the corresponding minor of the schema of coefficients, divided by the determinant Δ.*

δ) We shall now calculate the determinant Δ of the coefficients' schema. To that end, we set up its square by the law of multiplication of determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_1^2 + a_2^2 + a_3^2 & b_1a_1 + b_2a_2 + b_3a_3 & c_1a_1 + c_2a_2 + c_3a_3 \\ a_1b_1 + a_2b_2 + a_3b_3 & b_1^2 + b_2^2 + b_3^2 & c_1b_1 + c_2b_2 + c_3b_3 \\ a_1c_1 + a_2c_2 + a_3c_3 & b_1c_1 + b_2c_2 + b_3c_3 & c_1^2 + c_2^2 + c_3^2 \end{vmatrix}$$

where the columns of the first determinant are multiplied by those of the second. According to the formulas (3) this product determinant simply is

$$\Delta^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\,,$$

so that finally $\Delta = \pm 1$. In order to decide which sign to choose, we note that we have thus far used only the relation (2), which is satisfied equally in rotation and in reflection. Now, among all orthogonal transformations, *rotations* have the property that they *can be generated from the identical transformation* $x' = x$, $y' = y$, $z' = z$, *by continuous variation of the coefficients*, corresponding to a continuous movement of the coordinate system from the original to the new position. On the other hand, the substitution which we call, in general, reflection, arises by continuous modification of the inversion $x' = -x$, $y' = -y$, $z' = -z$, whereas this inversion itself cannot be generated continuously from the identical transformation. However, the determinant of the substitution is a continuous function of the coefficients, and it must change continuously when we change the identical transformation continuously into an arbitrary rotation. Its value at the start is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = +1 .$$

Since its value, as we have seen, is always either $+1$ or -1, it must of necessity
[46] remain always $+1$ for rotations, for an abrupt change to -1 would mean a discontinuity. Hence *for every rotation the determinant* Δ *has the value*

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = +1 , \tag{6}$$

and, similarly, for *every reflection*, we must have $\Delta = -1$.

The formula (5) now takes the simple form:

$$a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} . \tag{7}$$

Thus *each coefficient in the schema of rotation substitutions for the rectangular coordinate system is equal to the corresponding minor.*

The Transformation Formulas for Some Elementary Quantities

We come now to our real task, to find out how the coordinates of the spatial elementary configurations, the line segment X, Y, Z, L, M, N, the plane segment $\mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{P}$, and finally the space segment T, behave under the four kinds of change of the rectangular system of coordinates.

To write down all the formulas of transformation would take too much space, and it would also eventually become tedious. Therefore I shall mention only a few points that deserve special notice. First, I make the remark, which you can easily

verify, that in all formulas of transformation of the coordinates of a line segment, the first three coordinates X', Y', Z' in the new system are expressed in terms of X, Y, Z alone, and, in fact, as linear homogeneous functions of them. The quantities L, M, N do not enter. Thus, *according to the general principle already announced* (*p. [27]–[28]*) *the totality of the three quantities X, Y, Z must, in itself, determine a geometric configuration independent of the system of coordinates*. This is the *free vector*, which we have mentioned already (p. [32]). In the same way, the three coordinates $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ of the plane segment are transformed without regard to the fourth, $\mathcal{P}$, *so that they also have geometric significance independent of the coordinate system*. They represent the *free plane quantity* already mentioned (p. [32]).

We shall now find out, by special calculation, how the *coordinates of the free vector X, Y, Z*, behave under our transformations $(A_1), \ldots, (A_4)$ (p. [43]). For that purpose, we replace only in $X' = x_1' - x_2', \ldots$ the $x_1', \ldots$ by x, y, z, by means of the formulas (A_2), and we obtain at once the following formulas.

1. For *parallel translation*:

$$(B_1) \qquad X' = X\,, \quad Y' = Y\,, \quad Z' = Z\,.$$

2. For *rotation*:

$$(B_2) \qquad \begin{cases} X' = a_1 X + b_1 Y + c_1 Z\,, \\ Y' = a_2 X + b_2 Y + c_2 Z\,, \\ Z' = a_3 X + b_3 Y + c_3 Z\,. \end{cases}$$

3. For *inversion*: **[47]**

$$(B_3) \qquad X' = -X\,, \quad Y' = -Y\,, \quad Z' = -Z\,.$$

4. *For change of unit length*:

$$(B_4) \qquad X' = \lambda X\,, \quad Y' = \lambda Y\,, \quad Z' = \lambda Z\,.$$

Thus, *under translation of the coordinate system, the coordinates of the free vector remain unchanged*; *otherwise, however, they behave exactly like the point coordinates themselves*.

Let us compare with this the formulas of transformation for a *couple*, which we obtain from the formulas of transformation of the coordinates of a line segment by putting $X = Y = Z = 0$. Then, of course,

$$X' = Y' = Z' = 0\,,$$

and, for the turning moments with respect to the new axes, we get the following formulas.

1. For *translation*:

$$(C_1) \qquad L' = L\,, \quad M' = M\,, \quad N' = N\,.$$

2. For *rotation*:

$$(C_2)\qquad \begin{cases} L' = a_1 L + b_1 M + c_1 N\,, \\ M' = a_2 L + b_2 M + c_2 N\,, \\ N' = a_3 L + b_3 M + c_3 N\,. \end{cases}$$

3. For *inversion*:

$$(C_3)\qquad L' = L\,,\quad M' = M\,,\quad N' = N\,.$$

4. For *change of unit length*:

$$(C_4)\qquad L' = \lambda^2 L\,,\quad M' = \lambda^2 M\,,\quad N' = \lambda^2 N\,.$$

The coordinates of a couple are unchanged, therefore, by translation of the coordinate system, and by inversion; they behave, under rotation, like point coordinates; and they are multiplied by the factor λ^2 under change of the unit of length, i.e., they have the dimension 2 (of an area), whereas the free vector, like point coordinates, has the dimension 1.

The formulas (C_1), (C_3), (C_4) are derived without any difficulty; perhaps (C_2) needs some explanation. Indeed, with the aid of rotation formulas (A_2), we get

$$L' = \begin{vmatrix} y_1' & z_1' \\ y_2' & z_2' \end{vmatrix} = \begin{vmatrix} a_2x_1 + b_2y_1 + c_2z_1 & a_3x_1 + b_3y_1 + c_3z_1 \\ a_2x_2 + b_2y_2 + c_2z_2 & a_3x_2 + b_3y_2 + c_3z_2 \end{vmatrix}\,.$$

If we multiply out the last determinant, we get $3\cdot 3 + 3\cdot 3 = 18$ terms, of which three sets of two terms (e.g., $a_2x_1 \cdot a_3x_2 - a_3x_1 \cdot a_2x_2, \ldots$) cancel. The remaining twelve terms can be collected into the following sum of products of determinants:

$$L' = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \cdot \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} + \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \cdot \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\,.$$

[48] According to formula (7), for minors of a coordinate system for a rotation (p. [46]), the first factors are equal to a_1, b_1, c_1, while the second factors are L, M, N. Thus the formula given above for L' has been obtained. The two other formulas for M' and N' follow similarly.

Couple and Free Plane Quantity as Equivalent Configurations

As a third configuration, let us now consider the *free plane quantity*. Very simple calculations like those above, which I shall leave for you to carry out, lead to the result *that the components $\mathcal{L}, \mathcal{M}, \mathcal{N}$ of a free plane quantity transform, in all cases, just as do the coordinates L, M, N of a couple*.

For the sake of clearness, let us combine these results into a *little table*. It gives the transformed first coordinates, from which the others result by cyclic interchange.

	TRANSLATION	ROTATION	INVERSION	CHANGE OF UNIT LENGTH
Free Vector	X	$a_1X + b_1Y + c_1Z$	$-X$	λX
Couple	L	$a_1L + b_1M + c_1N$	L	$\lambda^2 L$
Free Plane quantity	$\mathcal{L}$	$a_1\mathcal{L} + b_1\mathcal{M} + c_1\mathcal{N}$	$\mathcal{L}$	$\lambda^2\mathcal{L}$

We have now obtained the precise foundation for a series of geometric statements, which appear in the textbooks frequently not at all, or only incidentally, and in a form in which their simple geometric content is not easily understandable. Often the geometric configurations, which we consider here are not at all separated in the clear cut manner, which we consider necessary, and, as a result, a whole series of interesting relations is completely obscured. For example, already with Poinsot, the concepts of couple ("couple") and free plane quantity ("aire"), from the start, are always tied together. Obviously this makes understanding necessarily more difficult. For us, a comparison of the last two lines of the above table shows, according to a general principle stated earlier, *that a couple and a free plane quantity are to be thought of as geometric fundamental configurations of the same sort*, because they behave in the same way under all changes of the rectangular coordinate system.

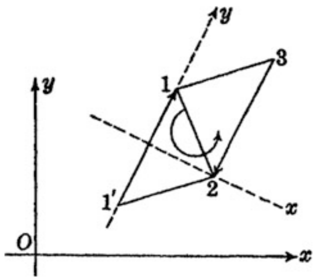

Figure 41

Let us make the content of this statement still clearer. If a couple L, M, N is given and we set up a relation between it and a plane quantity $\mathcal{L}, \mathcal{M}, \mathcal{N}$ by means of the equations $\mathcal{L} = L$, $\mathcal{M} = M$, $\mathcal{N} = N$ (or if we set it up in reverse order, starting from $\mathcal{L}, \mathcal{M}, \mathcal{N}$), then this coincidence remains unaffected by any transformation of coordinates. It must therefore be susceptible of pure geometric description without making use of a coordinate system. For this purpose, we start with the plane quantity $\mathcal{L}, \mathcal{M}, \mathcal{N}$, and specialise the coordinate system most conveniently by setting **[49]** $\mathcal{L} = \mathcal{M} = 0$. Then the free plane quantity represents a triangle $(1, 2, 3)$ lying in the x-y-plane or parallel to it, such that $\mathcal{N}$ is twice its area, i.e., equal to the area of the parallelogram $(1, 1', 2, 3)$, where the sign is to be determined by the circuit direction for $1\,1'\,2$ (see Fig. 41). I assert, now, that the corresponding couple, with the moments $\mathcal{L} = 0$, $\mathcal{M} = 0$, $N = \mathcal{N}$ can be formed with the opposite paral-

lelogram sides $(1, 1')$ and $(2, 3)$, with the arrow-heads at 1 and 2. To prove this, I choose the system of coordinates in the x-y-plane still more conveniently, namely, with the y-axis in the line $1\,1'$ and the x-axis through the point 2. (Drawn dotted in Fig. 41.) Then the two line segments $(1, 1')$ and $(2, 3)$, and likewise the couple formed by them, have the turning-moments $L = 0$ and $M = 0$. Moreover, the third turning-moment for the line segment $(1, 1')$ is also zero, so that finally N is equal to the turning-moment of $(2, 3)$:

$$N = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = x_2 \cdot y_3 ,$$

(for $x_2 = x_3$ and $y_2 = 0$, according to our assumption). On the other hand, for this position of the coordinate system, the third coordinate of the plane segment is

$$\mathcal{N} = \begin{vmatrix} 0 & y_1 & 1 \\ x_2 & 0 & 1 \\ x_2 & y_3 & 1 \end{vmatrix} = x_2 \cdot y_3 ,$$

that is, the product of the base y_3 of the parallelogram by the height x_2. Thus $N = \mathcal{N}$ in sign as well as quantity, which proves my statement.

We can state this as a general result, without reference to a special coordinate system. *A free plane segment*, *represented by a parallelogram of definite direction of the circuit*, *and the couple given by two opposite sides of the parallelogram*, *with arrows directed opposite to that direction*, *are geometrically equivalent configurations*, *i.e.*, *they have equal components with reference to every coordinate system.* Thus this theorem permits, at any time, the replacement of a couple by a parallelogram, or of the latter by a couple.

Free Vectors and Free Plane Quantities

We need pay no further attention to the second row of the table (p. [48]), and we
[50] shall compare the first and the third rows, i.e., the *free vector and the free plane quantity*. We notice, first, that both behave in the same manner, under translation and rotation, but that a difference appears when we add reflection or even a change of the unit of length. In order to follow this in detail, we think of a plane quantity $\mathcal{L}, \mathcal{M}, \mathcal{N}$, given in the familiar (right-handed) coordinate system, and we associate with it a free vector by means of the equations $X = \mathcal{L}$, $Y = \mathcal{M}$, $Z = \mathcal{N}$. These equations will remain unaffected if we restrict ourselves to movements of the system of coordinates, but they will be modified by reflection or by change of the unit of length. If we wish to give geometric expression to them, we cannot get along without taking account of the direction of the coordinate system and of the unit of length. In fact, if we again place the coordinate system as before, so that $\mathcal{L} = \mathcal{M} = 0$ and $\mathcal{N}$ is equal to the area of the parallelogram $(1, 1', 2, 3)$ in the x-y-plane, then, as the figure shows (see Fig. 42), $\mathcal{N} > 0$, and the vector $X = Y = 0$,

$Z = \mathcal{N}$ has the positive direction of the z-axis. Obviously, we can state this fact independently of the special position of the coordinate system: *In order to obtain, in a right-handed system of coordinates, the free vector which has the same coordinates as a given plane quantity, we erect a normal to the plane, toward that side from which the boundary of the parallelogram representing the plane quantity appears counterclockwise, and we lay off on it a segment equal to the area of the parallelogram.* The equality between the coordinates of the vector and of the plane quantity persists, no matter how one translates or rotates the coordinate system. It ceases, however, if we perform an inversion, or if we change the unit of length. For example, if we measure in decimetres, instead of in centimetres, the measure of the area is divided by 100, that of the vector segment only by 10; likewise, under inversion, the vector changes sign, but not the plane quantity.

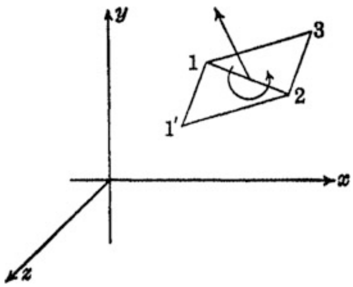

Figure 42

We can identify a free plane quantity completely with a free vector only if we choose once for all a definite direction for the coordinate system and a definite unit of length. Each person is free, of course, to impose such a restriction according to his whim, but he must recognise the arbitrary nature of his choice, if he would come to an understanding with others. All these things are, as you see, very clear and simple, but they must always be borne in mind because the historical development has left a certain confusion in present-day physics. A word, therefore, concerning **[51]** the *history of these matters*.

Scalars of First and Second Kind

Hermann Graßmann[22]'s theory of extension, of 1844, because it was so hard to read, as I have emphasised, made little impression upon our physics and mechanics. The development by William R. Hamilton in Dublin, at about the same time had much more influence in England. Hamilton was the inventor of *quaternions*, which I con-

[22] [Translator's note: One of Graßmann's sons became the same name, Hermann, and turned out to work as a mathematician, too, and thus one uses to distinguish both by "the Elder" and "Junior".]

sidered at length[23] during the winter semester. I need to add here only that he also introduced the word *vector* for what we have called *free vector*, whereas he did not expressly use the notion of *line-bound vector*. Furthermore, he did *not distinguish between free plane quantity and free vector*, because, at the outset, he assumed a definite determination of the coordinates as to meaning and as to scale of coordinates. This usage went over into physics, where, for a long time, no distinction was made between real vectors and plane quantities. To be sure, there arose gradually, in finer investigations, the need for a separation of two forms, according to their behaviour under inversion, both of which had been called indiscriminately *vector*, and for this purpose, the adjectives "*polar*" and "*axial*" were introduced. *A polar vector changes its sign under inversion*, *and is thus identical with our free vector*; *an axial vector does not change under inversion*, *and agrees*, *therefore*, *with our free plane quantity* (whereby we take no account of dimension). Eventually, physics had to recognise here a difference, which is surprising in some ways, and which occurs still in the usual presentations, but which, in our general treatment, appears from the start as quite natural.

Let us now give an example, which will clarify this discussion. The statement that electric excitation is a polar vector means that it is measured by three quantities X, Y, Z, which transform according to the first row of our table (p. [48]). The corresponding statement that the magnetic field strength is an axial vector means that its three components change according to the last row in the table. To be sure, I leave here undetermined the question as to the dimension of these components, as that would take us too far into physical details.

Along with the word *vector*, Hamilton introduced the word *scalar*, which also plays an important role in physics today. *A scalar is simply a quantity that is an invariant under all of our transformations of coordinates*, i.e., a quantity that, under changes of the coordinate system, itself changes either not at all, or only by a factor.
[52] If we go into detail, we can *distinguish different shades in the notion of scalar*. Let us consider, first, as example, the *space segment*, *or the volume of the tetrahedron*:

$$T = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

This transforms, as is easily verified by calculation, as follows:

under	TRANSLATION	ROTATION	INVERSION	CHANGE OF UNIT LENGTH
over into	T	T	$-T$	$\lambda^3 T$

Such a quantity, which is unchanged by translation or rotation, but is changed in sign by reflection, is called a *scalar of the second kind*, while a *scalar of the first*

[23] See volume I, pp. [64] sqq.

kind is unchanged also by inversion. The dimension, which is given by the fourth column, is not considered in this statement.

We can also easily set up *scalars of the first kind*. The simplest examples are $X^2+Y^2+Z^2$, where X, Y, Z are the coordinates of a free vector, and $\mathcal{L}^2+\mathcal{M}^2+\mathcal{N}^2$, where $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are the coordinates of a free plane quantity. That these quantities remain, in fact, unchanged by all movements and reflections (not by changes in the unit of length) can be inferred from the table on p. [48], if we also take into account equations (3), p. [44], for the coefficients of rotation. They must, therefore, have a pure geometric meaning. Indeed we know that they represent the square of the length of the vector, or, as the case may be, of the area of the plane segment.

We shall now inquire *how we can obtain, from combinations of given fundamental configurations* (*vectors and scalars of both kinds*), *additional configurations of the same species*. We shall consider first a very simple example. Let T be a scalar of the second kind, say the volume of a tetrahedron, and let X, Y, Z be the coordinates of a polar vector. We consider the three quantities $T\cdot X$, $T\cdot Y$, $T\cdot Z$. They transform, under movements, just as do the vector components X, Y, Z themselves. Under inversion, however, they remain unchanged, because both factors change sign. Thus these three magnitudes represent an axial vector. Similarly, starting with an axial vector $\mathcal{L}, \mathcal{M}, \mathcal{N}$ we can obtain a polar vector $T\cdot\mathcal{L}$, $T\cdot\mathcal{M}$, $T\cdot\mathcal{N}$.

Now we shall take *two polar vectors* X_1, Y_1, Z_1 *and* X_2, Y_2, Z_2 and we shall form from them all sorts of characteristic combinations, starting with a purely analytic procedure. We shall examine the behaviour of the newly formed quantities under transformation of coordinates and we shall decide from this what sort of geometric quantities they represent.

1. We start with the three sums X_1+X_2, Y_1+Y_2, Z_1+Z_2. They transform **[53]** in just the same way, obviously, as do the vector components themselves; hence they represent a new *polar vector*, which has with the two given vectors a purely geometric relation, which is independent of the coordinate system.

2. The bilinear combination of both vector components

$$X_1X_2+Y_1Y_2+Z_1Z_2$$

remains unchanged by all movements and reflections, as is easily verified by calculation; hence it represents a *scalar of the first sort*, which, as such, must admit a purely geometric definition.

3. The three minors of the matrix formed from the components

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

behave, as is easily shown, just as do the coordinates of *a free plane quantity or of an axial vector*, which must then be connected with the given vectors independently of the coordinate system.

4. We consider finally *three polar vectors*, and form out of their nine components the determinant

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}.$$

This remains unchanged under all movements, but it changes sign under reflection, so that it defines a *scalar of the second kind*.

I shall indicate the geometric interpretation of these configurations. After the result is once stated, you can easily complete the proofs, if you will only start from a properly specialised position of the coordinate system.

Interpretation of 1. The interpretation of the so-called *sum of the two vectors*, defined here, is well known. IE the two vectors are drawn from the same point, then *the diagonal*, drawn from that point, of the *parallelogram formed from them* represents this sum. [*Rule of the* "*parallelogram of forces*." (See Fig. 43.)]

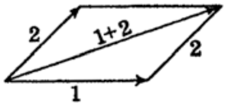

Figure 43

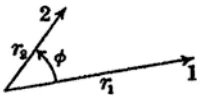

Figure 44

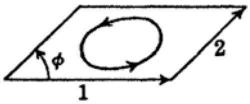

Figure 45

Interpretation of 2. If the vectors have the *lengths* r_1 *and* r_2, and if the *angle between their directions is* ϕ (see Fig. 44), then the *bilinear combination* $r_1 r_2 \cos\phi$.

Interpretation of 3. We consider, again, a *parallelogram*, *whose sides are paral-*
[54] *lel to the vectors* 1 *and* 2, *and we think of it as travelled around in the sense given by the succession of the directions of* 1 *and* 2 (see Fig. 45); then we have a completely determined free plane quantity, precisely the one defined above by its three coordinates. Moreover, *the absolute value* of its area is given by $r_1 \cdot r_2 |\sin\phi|$.

Interpretation of 4. If the three vectors all start from one point, they form the *three edges of a parallelepipedon* (see Fig. 46) whose volume, with properly determined sign, will be equal to the scalar of the second kind defined by the determinant.

Let me speak now of the way in which these processes appear elsewhere in the literature, where it is not customary to give primary importance, as we do here, to an investigation of the behaviour of certain analytic expressions under transformation of the coordinates, i.e., to a rational and simple theory of invariants. In the usual treatments, a certain nomenclature in mechanics and physics has been evolved, following Graßmann and Hamilton. It is customary to speak about the so-called vector algebra, and about vector analysis, which compares the rules of formation of new vectors and scalars from given vectors with the elementary rules of operation upon ordinary numbers.

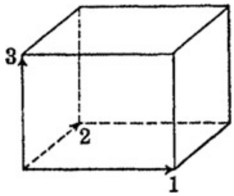

Figure 46

We first note that the operation appearing in No. 1 is called, as already indicated, simply the *addition of the two vectors* 1 *and* 2. Justification for this designation is found in the validity of certain formal laws, which characterise the addition of ordinary numbers, in particular, the *commutative and the associative laws*. The first of these laws states that the definition of the "sum" is independent of the order in which the two vectors 1 and 2 are used. The second of the two laws states that the addition of the sum of 1 and 2 to a vector 3 gives the same result as the addition of 1 to the sum of 2 and 3. In a much freer manner, the operations defined in No. 2 and in No. 3 have been called *multiplication*, and we distinguish between *inner or scalar multiplication* (No. 2) and *outer or vector multiplication* (No. 3). Indeed, in each of these, the important property called *the distributive law of multiplication with respect to addition*, which is expressed by the equation

$$a_1(a_2 + a_3) = a_1 a_2 + a_1 a_3\,,$$

is valid. In fact, for inner multiplication, we have

$$\begin{aligned} &X_1(X_2 + X_3) + Y_1(Y_2 + Y_3) + Z_1(Z_2 + Z_3) \\ &= (X_1X_2 + Y_1Y_2 + Z_1Z_2) + (X_1X_3 + Y_1Y_3 + Z_1Z_3)\,. \end{aligned}$$

The analogous property for outer multiplication can be derived with equal simplicity. **[55]** As to the other formal laws of multiplication, which I discussed fully in my lecture course[24] last winter, I may say that the commutative law ($a \cdot b = b \cdot a$) holds for inner multiplication, but not for outer multiplication, since the small determinants of the matrix which defines the outer product change sign when the vectors 1 and 2 are interchanged.

[24] See Volume I, p. [10].

I may add that the outer product of two polar vectors is often defined simply as a vector, without sufficiently emphasising its axial character. Of course, on the basis of the general relation given above (p. [50]), we can replace the free plane quantity by a vector, and we obtain the following rule. *The outer product of two vectors* 1 *and* 2 *is a vector* 3 *of length* $r_1 r_2|\sin\phi|$, *perpendicular to the plane of* 1 *and* 2, *and so directed that the vectors* 1, 2, 3 *are oriented to each other as are the positive* x, y, z *axes*, *respectively*, *to one another* (see Fig. 47). It must not be forgotten, however, that this definition depends essentially upon the kind of coordinate system and upon the unit of length.

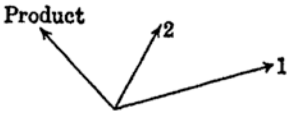

Figure 47

Missing Uniform Notation for Vector Calculus

Why this *language of vector analysis* has been so firmly adopted I am unable fully to understand. It may well have some connection with the fact that many people derive much pleasure from such formal analogies with the common time-honoured operations of reckoning. In any event, these names for the vector operations have been accepted with sufficient generality. However, the choice of a definite symbolism for these operations, in particular for the different kinds of multiplication, has resulted in a great divergence of opinion. In my preceding course of lectures,[25] I explained that there remains great disagreement, in spite of all efforts. Meanwhile, an international commission was set up at the recent mathematical Congress in Rome, and was asked to propose a unified notation. Whether any sort of agreement will be reached even among the members of this Commission, and whether the great body of mathematicians will accept its proposals, only time will tell. It is extraordinarily difficult to induce a large number of individuals, bent upon going comfortably in their own ways, to reconcile their divergent views, except under the compelling force of legislative enactment or of material interest. I prefer not to talk here about the notation of vector analysis; otherwise I might unwittingly create a new one.

[56] I do not wish to end this digression without pointing out, with emphasis, that, for our general standpoint, the *questions of ordinary vector analysis constitute only a chapter* out of a profusion of more general problems. For example, *line-bound vectors*, *restricted plane quantities*, *screws*, *and systems of forces* are, at first, not considered in vector analysis. For a real understanding of the operations of vector algebra themselves, however, it is actually necessary to take a broader view of them. Only then does the principle, which inheres in them, namely, that of defin-

[25] Volume I, p. [71].

ing geometric quantities according to their behaviour under the various kinds of transformation of rectangular coordinates, find full expression. As to the literature concerning all these questions, I mention first the work in which I explained our general principle of classification and applied it, in particular, to the above mentioned theory of screws: *Zur Schraubentheorie von Sir Robert Ball*.[26] I should mention also the *Enzyclopädie* reports by Heinrich Emil Timerding (*Geometrische Grundlegung der Mechanik eines starren Körpers*, Enz. IV, 2) and Max Abraham (*Geometrische Grundbegriffe der Mechanik deformierbarer Körper*, Enz. IV, 14).

[The Committee, which was set up in Rome for the unification of vector notation did not have the slightest success, as was to have been expected. At the following Congress in Cambridge (1912), they had to explain that they had not finished their task, and to request that their time be extended to the meeting of the next Congress, which was to have taken place in Stockholm in 1916, but which was omitted because of the war. The committee on units and symbols (called AEF) met a similar fate. It published in 1921 a proposed notation for vector quantities, which aroused at once and from many sides the most violent opposition. This plan is printed in volume I (1921) of the *Zeitschrift für angewandte Mathematik und Mechanik*, pp. 421–422. The comments of the opponents are published in the second volume (1922) of the same journal. The terminology which is usual today in vector calculation comes historically, in the main, from two sources, from Hamilton's quaternion calculus and from Graßmann's theory of extension. The developments of Graßmann were hard to read and remained unknown to German physicists; for a long time they formed a sort of esoteric doctrine for small mathematical groups. The ideas of Hamilton, on the other hand, made their way into English physics, mainly through James Clerk Maxwell. In his *Treatise on Electricity and Magnetism* (2 vols., Oxford, 1873), however, Maxwell used, in his vector equations, the representation by components almost exclusively. He made little use of a particular notation, through **[57]**
fear of not being understood, although in his opinion it was desirable, for many purposes in physical reflections, to avoid the introduction of coordinates and to draw attention instantly to a point in space instead of to its three coordinates, and to the direction and magnitude of a force rather than to its three components. That which today is called the vector calculus of the physicist is derived from the work of the telegraph engineer Oliver Heaviside and the American scholar Josiah W. Gibbs. The latter published in 1881 his *Elements of Vector Analysis*. Although Heaviside, as well as Gibbs, were Hamiltonians at the start, they both took over Graßmann's ideas into their calculus. Indirectly, through the works of these two authors, the vector calculus, and with it Graßmann's theory of extension, as well as Hamilton's quaternion calculus, made its way into German physics. The first book that introduced the vector calculus into the circle of German physicists, and that after the manner of Heaviside, was August Föppl's *Einführung in die Maxwell'sche Theorie*, which appeared in 1894. Both Graßmann and Hamilton had this in common, that the object of each was to operate with directed quantities, themselves, and only

[26] *Zeitschrift für Mathematik und Physik*, vol. 47, pp. 237 sqq., and *Mathematische Annalen*, vol. 67, p. 419 - F. Klein, *Gesammelte Mathematische Abhandlungen*, vol. 1, p. 503 et seq.

later to go over to their components. It is remarkable that both generalised the meaning of the word “product.” This may be due to the fact that, from the outset, they associate their developments with the theory of complex numbers of more than two terms. (See my presentation of quaternions in Vol. I, p. [64] sqq.) Otherwise, however, the technical terms of the two are entirely different, as has been shown already. The terms line segment, plane segment, plane quantity, inner and outer product, come from Graßmann, while the words scalar, vector, scalar product, and vector product, come from Hamilton. The disciples of Graßmann, in other ways so orthodox, replaced in part the appropriate expressions of the master by others. The existing terminologies were merged or modified, and the symbols, which indicate the separate operations have been used with the greatest arbitrariness. On these accounts, even for the expert, a great lack of clearness has crept into this field, which is mathematically so simple.

The principle exposed on p. [27] is a guiding star through this confusion. According to it, we can characterise the theories of Graßmann and Hamilton as follows. While Graßmann in his *Lineale Ausdehnungslehre* studies the theory of those invariants, which belong to the group of affine[27] transformations, which leave the origin of coordinates unchanged, he builds on the group of rotations in his later
[58] *Vollständige Ausdehnungslehre*, as does Hamilton also in his *Quaternions*. Hamilton’s procedure in this is thoroughly naive. It did not occur to him that there is anything arbitrary in the choice of the orthogonal group. Other differences can arise, as already explained, if inversion, that is, reflection of all the coordinate axes in the origin, is admitted on the one hand or is excluded as superfluous on the other. The whole situation can best be made clear with the notions *outer product* (free plane quantity), *vector product*, and *vector*. If we select the group of orthogonal transformations but exclude inversion, we make no distinction between these three types of quantity. For this reason, Graßmann, in his *Vollständige Ausdehnungslehre*, represents the free plane quantity (a parallelogram with a direction of rotation) by means of a vector, which he calls the complement of the plane quantity, and which corresponds completely to the vector which the physicist designates as a vector product. But if inversion is admitted, then “plane quantity” and “vector product” are to be considered equivalent geometric configurations, but different from that of “vector.” This corresponds to the customary distinction in physics between polar and axial vectors. If we now go over to the group of affine transformations, we can no longer consider Graßmann’s free plane quantity and the vector product as geometric quantities of the same kind.]

[27] These transformations are discussed later in this book (see pp. [75] sqq.).

V. Higher Configurations

Configurations of Points (Curves, Surfaces, Point Sets)

This completes what I wished to say here about elementary configurations of geometry, and I shall now turn to *the higher configurations, which arise by combination of these*. I shall do this in a *historical form*, so that you can get an idea of the development of geometry in the different centuries.

A. *Up to the end of the eighteenth century only points were commonly used as elementary configurations*. Other elementary configurations appeared at times, but never systematically. As configurations derived from points, there were considered *curves and surfaces* as well as more general *configurations* made up of parts of different curves and surfaces. Let us consider, briefly, how varied such configurations may be.

1. In *elementary instruction*, and sometimes even in the introductory course in analytic geometry, it would appear as though the whole of geometry were confined to the *straight line*, the *plane*, the *conic sections*, and *surfaces of the second order*. Of course that is a very narrow view. Even the knowledge of the ancient Greeks went beyond this, in part, for they included certain higher curves which they considered as "geometric loci." To be sure, these things had not reached down into **[59]**
ordinary instruction.

2. Let us compare with this the state of knowledge *around 1650*, when analytic geometry began with *Fermat* and *Descartes*. In those days, scholars distinguished between *geometric and mechanical curves*. The first type included particularly the conic sections, but included also certain higher curves such as those which are now called *algebraic curves*; the second type included such curves as those defined by some *mechanism*, e.g., the *cycloids*, which arise when a wheel rolls. Such curves belong for the most part to the curves now called *transcendental* curves.

3. Both these types of curves are included under *analytic curves*, which were defined later. These are curves whose *coordinates* x, y can be represented as *analytic functions of a parameter* t, i.e., briefly, as power series in t.

4. In recent times, consideration has often been given to *non-analytic curves*, whose coordinates $x = \phi(t)$, $y = \psi(t)$ cannot be developed into power series. Such are, for example, the curves defined by continuous functions without derivatives.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_6

This implies *a more general notion of curve, which includes the analytic curve as a special case*.

5. Finally, through the development, in recent times, of set theory, which I have discussed before,[28] a concept has appeared which was heretofore unknown, namely, the *infinite point sets*. This is a *totality of infinitely many points*, a point cluster, which may not exactly form a curve, but which is still defined by a certain law. If we wish to find, in our concrete perception, something that corresponds fairly well to a point set, we might look at the milky way, in which more careful search discloses ever more and more stars. The exact Infinite of the abstract point set theory is replaced here by the Infinite of the mathematics of approximation.

Within the scope of this course of lectures there will not be room, unfortunately, for more than this brief account of these disciplines, in particular for *infinitesimal geometry* and *point set theory*, although these are, of course, likewise important parts[29] of geometry. They are, however, treated thoroughly in many special lecture courses and books. Hence we shall give only this indication of their place in the whole field of geometry, in order that we may treat more fully things that are not so often treated elsewhere.

On the Difference Between Analytic and Synthetic Geometry

However, I should like to add to this account an explanation of the *difference*
[60] *between analytic and synthetic geometry*, which always plays a part in such discussions. According to their original meaning, synthesis and analysis are different methods of presentation. Synthesis begins with details, and builds up from them more general, and finally the most general notions. Analysis, on the contrary, starts with the most general, and separates out more and more the details. It is precisely this difference in meaning, which finds its expression in the designations synthetic and analytic chemistry. Likewise, in school geometry, we speak of the *analysis of geometric constructions*: we assume there that the desired triangle has been found, and we then dissect the given problem into separate partial problems.

In higher mathematics, however, these words have, curiously, taken on an entirely different meaning. *Synthetic geometry is that which studies figures as such, without recourse to formulas, whereas analytic geometry consistently makes use of such formulas as can be written down after the adoption of an appropriate system of coordinates*. Rightly understood, there exists only a *difference of gradation* between these two kinds of geometry, according as one gives *more prominence to the figures or to the formulas*. Analytic geometry which dispenses entirely with geometric representation can hardly be called geometry; synthetic geometry does not get very far unless it makes use of a suitable language of formulas to give precise expression to its results. Our procedure, in this course, has been to recognise

[28] See Volume I, pp. [271] sqq.

[29] Volume III will contain something about these issues.

this, for we used formulas from the start and we then inquired into their geometric meaning.

In mathematics, however, as everywhere else, men are inclined to form parties, so that there arose *schools of pure synthesists* and *schools of pure analysts*, who placed chief emphasis upon absolute "purity of method," and who were thus more one-sided than the nature of the subject demanded. Thus the analytic geometricians often lost themselves in blind calculations, devoid of any geometric representation. The synthesists, on the other hand, saw salvation in an artificial avoidance of all formulas, and thus they accomplished nothing more, finally, than to develop their own peculiar language formulas, different from ordinary formulas. Such exaggeration of the essential fundamental principles into scientific schools leads to a *certain petrifaction*; when this occurs, stimulation to renewed progress in the science comes principally from "outsiders." Thus, in the case of geometry, it was investigators in function theory who first made clear the difference between analytic and non-analytic curves, a difference which had never received sufficient attention either from the scientific representatives, or in the textbooks, of either of the two schools. Similarly, it was the physicists, as we have seen, who gave currency to vector analysis, although the fundamental notions are found in Graßmann. Even in texts on geometry today, vectors are often scarcely mentioned as independent concepts! **[61]**

From time to time, it has been proposed that geometry, as an independent teaching subject, be separated from mathematics, and that, generally speaking, mathematics, for purposes of teaching, be resolved into its separate disciplines. In fact, there have been created, especially in foreign universities, special professorships for geometry, algebra, differential calculus, etc. From the preceding discussion, I should like to draw the inference that the creation of such narrow limits is not advisable. On the contrary, the greatest possible living interaction of the different branches of the science, which have a common interest should be permitted. Each individual should feel himself, in principle, as representing mathematics as a whole. Following the same idea, I favour the most active relations between mathematicians and the representatives of all the different sciences.

With this, I am finishing this digression and I shall consider, following the historical development:

B. *the powerful impulse that geometric research received*, *from 1800 on*, when the so-called *newer geometry* stepped into the foreground. Today we call it, rather, *projective geometry*, because the operation of projection, which I shall discuss at length later, plays a chief role. The name "newer" is still used a good deal, but really inappropriately, because many still newer tendencies have appeared since then. As the first pathfinding researcher, I would name Jean-Victor Poncelet, whose *Traité des propriétés projectives des Figures*[30] appeared in 1822.

The difference between the synthetic and the analytic direction also played a role, from the first, in the further development of this projective geometry. As representing the first type, I mention the Germans Jacob Steiner and Carl Georg Christian von Staudt; among the second group, in addition to August F. Möbius, comes, above

[30] Second edition, Paris, 1865–66.

all, Julius Plücker. The fundamental works of these men, which have even today an active influence, are: Steiner's *Systematische Entwickelung der Abhängigheit geometrischer Gestalten von einander*,[31] von Staudt's *Geometrie der Lage*,[32] Möbius' *Baryzentrischer Kalkül*,[33] and, finally, Plücker's *Analytisch-geometrische Entwickelungen*.[34]

Projective Geometry and the Principle of Duality

If I were to indicate the most important guiding principles of these "newer" geometries, I should name first:

[62] 1. as the chief accomplishment of *Poncelet*, his giving prominence for the first time to the thought that there are *other configurations that have equal status with the point*. *In particular*, we may, in the plane, confront the *unlimited line* against the point, and in space confront the *unlimited plane* and the point. In a large number of the theorems in geometry, we can replace the Word "point" by the term "straight line" or by the word "plane," as the case may be. This is the *principle of duality*.

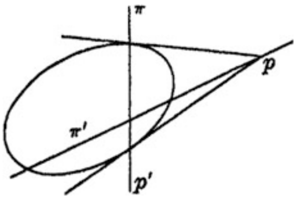

Figure 48

Poncelet connected his developments with the *theory of reciprocal polars*, the *polar theory of the conic sections*. As is well known, to every point p there belongs, with reference to a definite conic section, a straight line π, the polar of the point, which may be defined, perhaps, as the line joining the points of contact of tangents drawn from p to the conic section (see Fig. 48). Conversely, there belongs to every line π a pole p, and one has the reciprocal relation that if a point p' lies on π, then π', the polar of p', goes through p. From this special one-to-one relation between points and lines in the plane, which the conic section establishes, together with the analogous correspondence between points and planes in space, which is set up by a surface of the second order, Poncelet concluded *that one could "dualise" in this way all theorems of geometry which have to do only with properties of position*, *the mutual incidence of point and line*, *or of point and plane*. A famous example

[31] Berlin, 1832 = Gesammelte Werke, vol. I (Berlin, 1881), pp. 229 sqq. Reprinted in nos. 82, 83 of Ostwald's *Klassiker der exakten Wissenschaften*.
[32] Nürnberg, 1847.
[33] quoted p. [17].
[34] Two vols., Essen, 1828, 1831.

is the *theorem of Blaise Pascal*, concerning the hexagon inscribed in a conic section, which dualises into *Charles Brianchon's theorem* concerning the hexagon of tangents circumscribed about the conic.

2. As time went on, a *deeper understanding of the duality principle* led to its being detached from the theory of polars, and to its recognition as a *consequence of the peculiar constitution of projective geometry*. This beautiful systematisation appears first in the work of Joseph Diaz Gergonne and of Steiner. You need only read the preface of Steiner's *Systematische Entwickelungen*,[35] where he pictures in enthusiastic words how projective geometry first brought order into the chaos of geometric theorems, and how everything arranges itself so naturally in it.

As I shall often have occasion to speak of this discipline in the course of these lectures, I should like to give a *brief survey* of it now. The principle of duality may be stated as follows. In the *fundamental notions and the fundamental theorems* (*axioms*) *of geometry*, *the point and the plane*, *in space*, *or the point and the line*, *if we restrict ourselves to the plane*, *always enter symmetrically*, i.e., these axioms, and hence the theorems logically derived from them, are dual by pairs. The so- **[63]**
called "measure relations" [i.e. metrics] of elementary geometry, such as distance, angle, etc., do not, in the first instance, appear at all in this discipline. We shall see, later, how they can be fitted in supplementarily. In detail, the composition of the structure is as follows.

(a) *Three kinds of configurations* are used as the simplest ones for a foundation: *the point*, *the* (*unlimited*) *straight line*, *the* (*unlimited*) *plane*.

(b) The following relations (called *theorems of connection or axioms of connection*) exist between these fundamental configurations: *Two points determine a line*, *three non-collinear points determine a plane*; *two planes determine a straight line*; *three non-collinear planes determine a point*. The *unrestricted validity* of these axioms will be brought about by the skilful introduction of improper (infinitely distant) elements in a way to be explained later.

(c) We now construct the *linear fundamental configurations* (i.e., those, which are defined analytically by linear equations).

I. The *fundamental configurations of the first kind*, each consisting of ∞^1 elements:

(α) The totality of points on a straight line: a *rectilinear point range*.

(β) The totality of planes through a straight line: *an axial family of planes*,

(γ) The straight lines through a point in a plane: a (*plane*) *family of straight lines*.

II. *Fundamental configurations of the second kind*, each consisting of ∞^2 elements:

(α) The plane as locus of its points: a *field of points*,

(α') The plane as locus of its straight lines: *& field of straight lines*.

(β) The planes through a fixed point: a *family of planes*.

(β') The lines through a fixed point: a *family of straight lines*.

[35] Cited above, p. 233.

III. *Fundamental configurations of the third kind*, each consisting of ∞^3 elements:

(α) Space as the locus of its points: a *space of points*.

(β) Space as the locus of its planes: a *space of planes*.

In this entire structure, complete duality appears everywhere. We can exhibit the whole body of projective geometry in two mutually dual ways if, using the given fundamental elements, we start on the one hand from points, and on the other either from straight lines, if we are concerned with geometry of the plane, or from planes if we are thinking of geometry of space.

Plücker's Analytical Conception and the Development of the Duality Principle (Straight Line Coordinates)

3. This structure can be represented in another manner, and more conveniently, if we follow the *analytic way* and inquire, for that purpose, in the first place, in what form the *principle of duality appears with Plücker*. We can write the *equation of the straight line* in the plane, if the constant term is not zero, as follows:

$$ux + vy + 1 = 0\,.$$

The straight line is determined if we know the values of the coefficients u and v, which, moreover, appear in this form symmetrically with the variable coordinates x
[64] and y. Now it is Plücker's conception to look upon these u and v as the "*coordinates of the line*" *and as having equal status with the point coordinates x and y*, *and as being considered*, *at times*, *as variable instead of them*. With this new point of view, x and y have fixed values, and the equation expresses the condition that a variable straight line passes through a fixed point: it is the equation of this point in straight line coordinates. Finally, one does not need to prefer one of these two terminological modes: one can leave it entirely undetermined which pair of quantities we will consider as constant and which as variable. Then the equation expresses the *condition for the* "*united position*" *of point and straight line*. *Now the principle of duality resides in the fact*, *that every equation in x and y*, *on one hand*, *and in u and v on the other hand*, *is completely symmetrical. Everything that we said above concerning the duality that is inherent in the axioms of connection resides in this property*.

In space, of course, the equation of the straight line will be replaced by the *equation of the plane*

$$ux + vy + wz + 1 = 0\,.$$

As a result of these considerations, geometry can be developed analytically by looking upon either x, y, z or u, v, w as the fundamental variables and, accordingly, by simply interchanging the words point and plane. In this way, then, arises the familiar *double construction of geometry*, which you find emphasised in many textbooks, where dual theorems appear side by side on the same page, separated by

a vertical line. Let us cast a rapid glance at the *higher configurations,* which arise in this way, always in *dual pairs*, whereby we shall, in a sense, obtain a continuation of the above dual scheme of linear configurations.

To start with, we look upon x, y, z as *definite*, *non-constant functions* ϕ, χ, ψ *of a parameter* t. These three functions will then represent a *space curve*, which, in particular (when ϕ, χ, ψ satisfy identically a linear equation with constant coefficients), can be a *plane curve*, or, finally (when they satisfy two such linear equations), can degenerate into a *straight line*. In the same way, considering u, v, w as functions of t, we obtain a *singly infinite succession of planes*, which we can consider most conveniently as the tangent planes enveloping a *developable surface*. As special cases we get, as the first case, that all the planes pass through one point, i.e., that they envelop a cone, and, as the second case, that they all go through *a fixed straight line*.

Secondly, if we consider x, y, z as *functions of two parameters* t and t', we get a *surface*, which, in particular, can degenerate into a *plane*. The dual of this is the *double infinity of planes enveloping a surface*, which can degenerate into a *family of planes* through a point.

Let us collect these results into a table: **[65]**

$\left.\begin{aligned} x &= \phi(t) \\ y &= \chi(t) \\ z &= \psi(t) \end{aligned}\right\}$	Curve (*Plane curve*) (*straight line*)	$\left.\begin{aligned} u &= \phi(t) \\ v &= \chi(t) \\ w &= \psi(t) \end{aligned}\right\}$	Developable surface (*Cone*) (*straight line*)
$\left.\begin{aligned} x &= \phi(t, t') \\ y &= \chi(t, t') \\ z &= \psi(t, t') \end{aligned}\right\}$	Surface (*Plane*)	$\left.\begin{aligned} u &= \phi(t, t') \\ v &= \chi(t, t') \\ w &= \psi(t, t') \end{aligned}\right\}$	Surface (*Point*)

This will suffice as an example of a dual scheme which men have found pleasure in developing, these many years.

4. One finds already in Plücker an *essential extension of this entire approach*. Just as he looked upon the three coefficients in the equation of the plane as variable plane coordinates, so he conceived the notion of considering, *quite generally*, *the constants upon which any geometric configuration depends* – e.g., the nine coefficients in the equation of a surface of order two – *as variable coordinates of this configuration*, *and he investigated what an equation between them might signify*. Of course, one can no longer talk of "duality," in any proper sense, since this depended upon the special property that the equation of the plane, as well as that of the straight line (see p. [63]), was symmetrical in coefficients and coordinates.

Plücker himself carried out this idea especially for *straight lines in space*. A straight line in space is given, in point coordinates, by two equations, which Plücker writes in the form

$$x = rz + \rho, \quad y = sz + \sigma .$$

The *four constants* r, s, ρ, σ in these equations are to be called the *coordinates of the straight line in space*. It is easy to show how they are related to the six ratios

$X : Y : \ldots : N$, derived by Graßmann's principle from two points of the straight line, which we have used before (pp. [32] sqq.). Plücker now considers an *equation* $f(r, s, \rho, \sigma) = 0$ between the four coordinates. It separates out from the four-fold infinity of straight lines in space a three-fold infinity of lines which Plücker calls a *line complex*. We have already mentioned (pp. [37] sqq.) the simplest case of this manifold, the *linear complex*. *Two equations*

$$f(r, s, \rho, \sigma) = 0\,, \quad g(r, s, \rho, \sigma) = 0$$

determine a *line-congruence*, which is also called a *system of rays*. The first of these names implies that we are concerned with those straight lines in which the two complexes $f = 0$, $g = 0$ coincide. Finally, *three equations* of the same sort, $f = g = h = 0$ determine a simply infinite family of straight lines, which cover a certain surface called a *ruled surface*.

[66] Plücker gave this presentation in 1868–69 in his book entitled *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*.[36] He died as the printing of the first part was nearly finished, and I, as his assistant, was able to win my spurs by editing the second part.

Plücker's general principle of considering any configuration as a space element and its constants as coordinates has led to other interesting developments. Thus the eminent Norwegian mathematician Sophus Lie, who worked many years in Leipzig, had great success with his *geometry of the sphere*. Here the space element is the sphere, which, like the straight line, depends upon four parameters. I mention, further, Eduard Study's *Geometrie der Dynamen*,[37] of a later date, where a whole series of interesting investigations of this nature are connected with the notion of the "dyname," which we have discussed above.

Graßmann's Theory of Extension; Higher Dimensional Geometry

C. The "new geometry" which we have been discussing is based primarily on the prominence given to the unlimited straight line and the unlimited plane as space elements. Graßmann's developments, beginning in 1844, went beyond this, however. Here he placed the *limited line segment*, *plane segment*, *space segment* in the foreground and assigned components to them according to his "*determinant principle"*, all of which we have discussed thoroughly. The beautiful thing about this is that it corresponds to the needs of mechanics and physics far more effectively than do, for example, line geometry and the principle of duality.

Of course, these different directions are by no means so sharply separated from one another as I have made it appear in my attempt to give you a clearer view of each of them. The fact of the matter is that Plücker gives *more* weight to the unlimited

[36] Parts 1, 2, Leipzig, 1868 and 1869.
[37] Leipzig, 1903.

straight line, Graßmann *more* to the line segment, while, with each of them, the other configuration sometimes appears. In particular, Study might just as well be placed in the present rubric as in the preceding one.

I must emphasise, however, that Graßmann by no means confined himself to concepts that were immediately applicable, but that, with unfettered creative instinct, he went far beyond that. His most important contribution is that he introduced the general notion of *n point coordinates* $x_1, x_2, \ldots, x_n$, instead of the three x, y, z, and so he became the real *creator of geometry of space*, R_n, *of n dimensions*. Following his general principle, he considered, in such a higher space, the matrices of the coordinates of $2, 3, \ldots, n+1$ points, whose minors then gave him a whole series of fundamental configurations of R_n, corresponding to the line segment and the plane segment. I have mentioned already that Graßmann called the abstract discipline **[67]** thus created the theory of extension.

This notion of R_n has been extended in recent times to include the consideration of *infinitely many coordinates* $x_1, x_2, \ldots$ *ad infinitum*, and one speaks of *space* R_∞ *of infinitely many dimensions*. That such a notion can make sense can be seen if we think of operating with power series: a power series is determined by the totality of its infinitely many coefficients, and it can, to that extent, be represented by a point in R_∞.

The strange issue here, as has been recognised in general by mathematicians, is that this way of speaking geometrically of n and, indeed, of infinitely many variables, has proved to be of real use. By means of it, discussions become more vivid than when they are confined to abstract analytic expression. The student acquires soon such facility in the use of the new geometric representation as to make it appear that he is really at home in R_n or R_∞. What measure of truth lies behind this phenomenon, and whether, perhaps, a natural gift of the human mind comes to light, which is ordinarily limited in its development by experience in space of only two or three dimensions – that is a question to be decided by psychologists and philosophers.

If I am to give you an orientation regarding the role of mathematics in general culture, I must devote a word to the turn, which was given to geometry of higher dimensions in 1873 by the astronomer Carl Friedrich Zöllner of Leipzig. We have here one of the rare cases where a mathematical expression has gone over into everyday use. Nowadays everybody uses expressions involving the "fourth dimension." This *popularizing of the fourth dimension* arose from experiments made before Zöllner by the spiritualist Slate [Henry Slade]. Slate announced himself as a medium that had direct intercourse with the spirits, and his exhibitions consisted, among others, in causing objects to disappear and to reappear. Zöllner believed in these experiments and set up for their explanation a physico-metaphysical theory, which was widely accepted. He postulated that for the real physical phenomenon, there is really a space of four or more dimensions, of which we, because of our limited endowment, can appreciate only a three-dimensional section $x_4 = 0$. He argued that an especially gifted medium that, perhaps, is in touch with beings living outside this world of ours, can remove objects from it, which would then become invisible to us, or he can bring them back again. He attempts to make these rela-

tions clear by picturing beings who are restricted to a two-dimensional surface, and whose perceptions have this limitation. We may think of the mode of life of certain
[68] animals, e.g., mites. If an object is removed from the surface in which these creatures live, it would appear to them to disappear entirely (that is how it is conceived), and it was in analogous fashion that Zöllner explained Slate's experiments. Various attempts have been made to picture the existence of these two-dimensional beings. Especially amusing is the one in an anonymous English booklet *Flatland*.[38] Here the author paints exactly the appearance of a two-dimensional world: the individual beings differ through their geometric form, being more complicated the more highly organised they are. Regular polygons are the highest beings. Women, of whom the author seems to have a poor opinion, have simply the form of a dash; and so it goes.

I hardly need to add here that the mathematical conception of geometry of higher dimensions has nothing to do with Zöllner's metaphysical notions. Mathematics shows itself here as a *pure normative science*, to use a modern expression, which considers the formally possible connections of things, and which exists quite independently of the facts of natural science or of metaphysics.

Scalar and Vector Fields; Rational Vector Analysis

After this digression, I should like to consider, in somewhat more detail, the higher configurations which, as *combinations of Graßmann's elementary configurations*, in particular of vectors, can be placed alongside of the combinations of points, planes, etc., which we have already discussed. We come here to the further *organisation of vector-analysis* proper, which, thanks especially to Hamilton, has become one of the most valuable tools of mechanics and physics. I place before you Hamilton's *Elements of Quaternions*, as well as the *Vector Analysis*,[39] already mentioned (p. [57]) by the likewise distinguished American J. W. Gibbs.

The new notion which is added here to our already familiar concepts of vector and scalar, is the *connecting of these quantities with the points of space*: To every point in space we assign a definite scalar $S = f(x, y, z)$ and we speak then of a *scalar field*. On the other hand, we attach to every point in space a definite vector

$$X = \phi(x, y, z)\,, \quad Y = \psi(x, y, z)\,, \quad Z = \chi(x, y, z)$$

and we call the totality of these vectors a *vector field*.

[69] In this way we designate two of the most important geometric notions, which are used everywhere in modern physics. It will suffice if I recall a few examples

[38] *A Romance of Many Dimensions*. By a Square. London, 1884. Basically, the purpose of the author here is to make comprehensible the possibility of a geometry of higher dimensions. [Translator's note: The author, publishing under this pseudonym, was Edwin Abbott Abbott, professor at Cambridge University.]
[39] Edited by E. B. Wilson, New York, 1901.

of their wide application. The density of a *mass distribution*, the *temperature*, the *potential energy* of a continuous extended system, always conceived of as a function of position, are examples of scalar fields. The *field of force*, in which a definite force is applied at each point, is the typical example of a vector field. I will cite the following additional examples. In the theory of elasticity, *the field of displacements* of a deformed body, when we assign to each point a line segment that indicates the amount and direction of its displacement, is a vector field. Similarly, in hydrodynamics, the *field of velocities*, and finally, in electrodynamics, the *electric and magnetic field*, in which to each point is assigned a definite electric and a magnetic vector, are examples of vector fields. Since at every point we can combine the vector of the magnetic field strength, which is of axial nature, with the polar vector of the electric field strength, to form a screw, the electromagnetic field can be interpreted also as an example of a *screw-field*.

Hamilton showed how these fields could be made available in the simplest way, for the methods of differential and integral calculus. To this end, it is fundamental to remark that the differentials dx, dy, dz, whose ratios determine the direction of displacement at a point of space, represent a *free vector*, i.e., *that they behave, under transformation of coordinates, as do free vector components*. This follows easily from the fact that they arise by a limit process from the coordinates of a small linear segment passing through the point x, y, z.

More important, but more difficult to grasp, is a second remark that *the symbols of partial differentiation*

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}$$

also have the character of free vector components, i.e., if we go over to a new rectangular coordinate system x', y', z', the new symbols $\partial/\partial x'$, $\partial/\partial y'$, $\partial/\partial z'$ behave toward the old as do the transformed coordinates of a vector (and, specifically, a *polar vector*).

This will be clear, at once, if we carry it out for a rotation of the coordinate system:

$$(1) \qquad \begin{cases} x' = a_1 x + b_1 y + c_1 z \,, \\ y' = a_2 x + b_2 y + c_2 z \,, \\ z' = a_3 x + b_3 y + c_3 z \,. \end{cases}$$

As we showed earlier extensively (p. [44]), these formulas of rotation have the **[70]** characteristic that their solution is obtained simply by the interchange of rows with columns in the system of coefficients:

$$(2) \qquad \begin{cases} x = a_1 x' + a_2 y' + a_3 z' \,, \\ y = b_1 x' + b_2 y' + b_3 z' \,, \\ z = c_1 x' + c_2 y' + c_3 z' \,. \end{cases}$$

If we have, now, any function of x, y, z, we can, by means of (2), express it as a function of x', y', z', and we shall have, according to the known rules for partial

differentiation,

$$\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x'}\,,\\
\frac{\partial}{\partial y'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y'}\,,\\
\frac{\partial}{\partial z'} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial z'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z'} + \frac{\partial}{\partial z}\frac{\partial z}{\partial z'}\,.
\end{aligned}$$

The derivatives of x, y, z with respect to x', y', z' are immediately available from (2), and we get

$$\begin{aligned}
\frac{\partial}{\partial x'} &= a_1\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y} + c_1\frac{\partial}{\partial z}\,,\\
\frac{\partial}{\partial y'} &= a_2\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + c_2\frac{\partial}{\partial z}\,,\\
\frac{\partial}{\partial z'} &= a_3\frac{\partial}{\partial x} + b_3\frac{\partial}{\partial y} + c_3\frac{\partial}{\partial z}\,.
\end{aligned}$$

A comparison with (1) shows, in fact, agreement with the transformation formulas for point coordinates, and thus for vector components.

An essentially simpler calculation would show also that, under *translation* of the system of coordinates, the three symbols $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ are unchanged, but that, under *inversion*, the sign changes, so that the statement is proved. To be sure, we have taken no account of changes in the unit of length, i.e., of dimension. If we do this, we find that our symbols have the dimension -1, because of the differentials of coordinates that appear in the denominators.

We shall now perform, with this Hamilton vector symbol ($\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$), the same operations that we performed earlier with vectors. Let me remark, in advance,
[71] that we may call the result of the operation $\partial/\partial x$ upon a function $f(x, y, z)$, that is, $\partial f/\partial x$, symbolically, *the product of* $\partial/\partial x$ *and* f, since the formal laws of multiplication, insofar as we are here concerned with them, in particular the distributive law

$$\frac{\partial(f+g)}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\,,$$

hold for these combinations.

Now let a *scalar field* $S = f(x, y, z)$ be given, and let us multiply this scalar by the components of the Hamilton vector symbol, in the sense just outlined, i.e., let us form the vector

$$\frac{\partial f}{\partial x}\,,\quad \frac{\partial f}{\partial y}\,,\quad \frac{\partial f}{\partial z}\,.$$

We have already seen (p. [48]) that the product of a scalar by a vector is again a vector. Since, in the proof of this theorem, only such properties of multiplication are used as persist also in our symbolic multiplication, it follows that *these three*

partial derivatives of the scalar field define a vector which depends on x, y, z and is thus a vector field. The connection between this vector field and the scalar field is independent of the particular coordinate system chosen. This vector field, with the sign changed, is called the *gradient of the scalar field*, a term taken from meteorology. Thus, in the familiar weather charts of the newspapers, the air pressure at each point is indicated as a scalar field S, while the curves $S =$ const, are drawn and the corresponding values of S are indicated. The gradient gives, then, the direction of the most rapid drop in air pressure and is always normal to these contour curves. One can always form a scalar $X^2 + Y^2 + Z^2$ from the vector components X, Y, Z. Hence we can obtain, from the gradient of a scalar, *a new scalar field*:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 ,$$

which must be connected with it, and therefore with the original scalar field, in a manner independent of the system of coordinates. This scalar is equal to the *square of the length of the gradient*, or, as it is called, to the *square of the slope of the scalar field f.*

Applying this same principle, we can form, from the vector symbol $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, a symbolic scalar, by multiplying symbolically each component by itself, i.e., by applying the operation which it implies twice. This yields the *operation*

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} ,$$

which has, *thus*, *scalar character*, i.e., it is invariant under transformation of coordinates. If we "multiply" this scalar symbol by a scalar field f, we get, necessarily, again a *scalar field* **[72]**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} ,$$

whose relation to the first one is independent of the coordinate system. If we think of a liquid flowing in a field, whose initial density is 1, and whose velocity at each point is given by the gradient of f, then the density at each point increases, in the first instant of time dt, by an amount equal to this scalar multiplied by dt. Hence we call

$$-\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

the *divergence of the gradient of f*.

Formerly, following Gabriel Lamé, it was customary to call a scalar field $S = f(x, y, z)$ also a *point function* (*fonction du point*), and to call the first scalar field connected with it, $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + (\partial f/\partial z)^2$, *the first differential parameter* and the second, $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$, the second differential parameter.

In similar manner, we shall now combine our vector symbol with a given (*polar*) *vector field*:

$$X = \phi(x, y, z) , \quad Y = \chi(x, y, z) , \quad Z = \psi(x, y, z) .$$

Indeed, we shall do this with the aid of both kinds of multiplication of two vectors with which we have become acquainted:

(a) By *inner multiplication* there results a scalar, which, in the already familiar notation of symbolic multiplication, may be written in the form:

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} .$$

Since this result also depends, of course, of x, y, z, it also represents a *scalar field* whose relation to the given vector field is independent of the system of coordinates. It is called, in the sense defined above, the *divergence* of that field.

(b) *Outer multiplication* yields the matrix:

$$\left| \begin{matrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{matrix} \right| ,$$

whose three determinants are to be read as:

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} , \quad \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} , \quad \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} .$$

[73] These define, according to what precedes, a *plane quantity*, or, as the case may be, an *axial vector* or an *axial vector field*. The connection between the two vector fields is again independent of the choice of the coordinate system. According to Maxwell, this vector field is called the *curl* of the given one. In Germany, the German word *quirl*, of like germanic origin, is used. Occasionally, this is called also *rotor*, or *rotation*.

We have thus obtained, through *systematic geometric investigation*, all those quantities, which the physicist must always have at hand in his study of the various vector fields. It is *pure geometry*, however, that we are studying. I must emphasise this all the more, since these things are often regarded as belonging to physics, and are therefore discussed in books and lecture courses on physics, instead of in geometry. In the nature of the case, such an attitude is thoroughly unjustified, and it is comprehensible only as a residue of the historical development. When the time came, physics had to create the instruments, which it needed, and which it did not find ready at hand in mathematics.

There exists here the same *misunderstanding*, which I mentioned often last semester in the field of analysis. In the course of time, physics developed all sorts of mathematical needs. Hence it often created valuable stimulation to mathematical science. But mathematics teaching, especially as it is given in the schools, even today, pays no attention to these changes. It goes along in the same old rut which it has followed for centuries, and leaves it to physics laboriously to provide its own aids, although these would supply much more appropriate material for mathematics teaching than do the traditional topics. You observe that in the life of the Intellect there is also a law of inertia. Everything continues to move along its old rectilinear

path, and every change, every transition to new and modern ways, meets strong resistance.

With this I leave the first main part, which has taught us the various kinds of geometric manifolds, the *objects of geometry*. Now we shall concern ourselves with a particular *method*, which is of greatest importance for the more exact study of these manifolds.

Second Part: Geometric Transformations [74]

General Remarks About Transformations and Their Analytic Representation

That which we now undertake is one of the most important chapters of scientific geometry. In its fundamental ideas and in its simpler portions it offers, however – and I wish especially to point this out in this lecture course – very stimulating material for school teaching. Geometric transformations are, after all, nothing more than a *generalization of the simple notion of function*, which our modern reform tendencies are striving to make the central point of mathematics teaching.

I begin with a discussion of *point transformations*, which constitute the simplest class of geometric transformations. They let the point persist as a space element, i.e., they bring every point into correspondence with another point – in contrast with other transformations which carry the point over into other space elements, such as the straight line, the plane, the sphere, etc. Here again I place the *analytic treatment* in the foreground, since it often enables us to give the most accurate expression of the facts.

The analytic expression of a point transformation is what analysis calls the *introduction of new variables* x', y', z':

$$\begin{cases} x' = \phi(x, y, z)\,, \\ y' = \chi(x, y, z)\,, \\ z' = \psi(x, y, z)\,. \end{cases}$$

We can interpret such a system of equations geometrically in two ways, I might say *actively* and *passively*. *Passively*, *it represents a change in the coordinate system*, i.e., the new coordinates x', y', z' are assigned to the point with the given coordinates x, y, z. This is the meaning we have always had in mind previously in our study of the changes of the rectangular system of coordinates. For general func-

tions ϕ, χ, ψ, these formulas include, of course, over and above that, the transition to other kinds of coordinate systems, e.g., trilinear coordinates, polar coordinates, elliptic coordinates, etc.

In contrast with this, the *active* interpretation holds the coordinate system fixed
[75] and changes space. To every point x, y, z, the point x', y', z' is made to correspond, so that there is, in fact, a transformation of the points in space. It is with this conception that we shall be concerned in what follows.

We shall obtain the *first examples* of point transformations, in the sense of these remarks, if we consider again the formulas which, before (see pp. [43]–[44]), *passively* interpreted, represented a translation, a rotation, a reflection, or a change in the unit of length, and we shall now interpret them *actively*. It is easy to see that the first two of these groups of formulas represent a *translation of space* – thought of as rigid – and a *rotation about O*, respectively, with respect to the immovable system of coordinates. The third group gives an *inversion of the points of space* in the origin O. [Every point x, y, z goes into $-x, -y, -z$, symmetric to it with respect to O (see Fig. 49).] The last one represents a so-called *similarity transformation of space*, with O as centre.

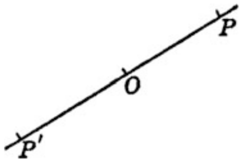

Figure 49

We now start our proper investigations with a particularly simple group of point transformations, which includes all the foregoing as subcases, namely, the *affine transformations*.

I. Affine Transformations

Analytic Definition and Basic Properties

An affine transformation is defined analytically when x', y', z' *are linear integral functions of* x, y, z:

$$
\left\{
\begin{aligned}
x' &= a_1 x + b_1 y + c_1 z + d_1 \, , \\
y' &= a_2 x + b_2 y + c_2 z + d_2 \, , \\
z' &= a_3 x + b_3 y + c_3 z + d_3 \, .
\end{aligned}
\right. \tag{1}
$$

The name, which goes back to *Möbius* and *Leonhard Euler*, implies that, in such a transformation, infinitely distant points correspond again to infinitely distant points, so that, in a sense, the "ends" of space are preserved. In fact, the formulas show at once that x', y', z' become infinite with x, y, z. This is in contrast to the general projective transformations, which we shall study later, in which x', y', z' are fractional linear functions, and by which, therefore, certain finite points will be moved to infinity. These affine transformations play an important role in physics under the name of *homogeneous deformations*. The word "homogeneous" implies (in contrast to heterogeneous) that the coefficients are independent of the position in space under consideration; the word "deformation" reminds us that, in general, the form of any body will be changed by the transformation.

The transformation (1) can be composed of displacements, in amounts d_1, d_2, d_3, **[76]** parallel to the three coordinate axes, together with the homogeneous linear transformation

$$
\left\{
\begin{aligned}
x' &= a_1 x + b_1 y + c_1 z \, , \\
y' &= a_2 x + b_2 y + c_2 z \, , \\
z' &= a_3 x + b_3 y + c_3 z \, ,
\end{aligned}
\right. \tag{2}
$$

which leaves the position of the origin unchanged (centro-affine transformation), and which is somewhat more convenient to be studied. We start the consideration of this type (2).

1. We inquire about the possibility of solving the system of equations (2). As the theory of determinants shows, this depends essentially upon whether the *deter-*

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_7

minant of the system of coefficients of the transformation

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{3}$$

vanishes or not. We shall consider the first case later; *for the present we shall assume that* $\Delta \neq 0$. Then (2) *has a unique solution of the form*

$$\begin{cases} x = a_1'x' + b_1'y' + c_1'z' \,, \\ y = a_2'x' + b_2'y' + c_2'z' \,, \\ z = a_3'x' + b_3'y' + c_3'z' \,, \end{cases} \tag{4}$$

where $a_1', \ldots, c_3'$ are the minors of Δ, divided by Δ itself. Thus to each point x, y, z we may say not only that there corresponds one point x', y', z', but also that there is *only one*, and the transition from x', y', z' to x, y, z is again an affine transformation.

2. We now ask *how the configurations in space change under these affine transformations*. To begin with, let us take a plane

$$Ax + By + Cz + D = 0\,,$$

substituting the expressions (4) for x, y, z, as equation for the corresponding configuration, we obtain

$$A'x' + B'y' + C'z' + D' = 0\,,$$

where the $A', \ldots, D'$ are certain combinations of $A, \ldots, D$ and of the coefficients of the transformation. In view of (1), we see that *every* point of the second plane arises from an appropriate point of the first. *Thus to every plane there corresponds another plane*. Since a straight line is the intersection of two planes, it *follows necessarily that to every straight line there corresponds another straight line*. Möbius calls transformations that have this property *collineations*, since they express the "collinearity" of three points, i.e., the property of lying upon a line. Hence *an affine*
[77] *transformation is a collineation*. If we investigate in the same way a *surface of the second degree*

$$Ax^2 + 2Bxy + Cy^2 + \ldots = 0\,,$$

using equations (4) to replace x, y, z by x', y', z', we obtain a quadratic equation. Hence *an affine transformation transforms every surface of second degree into another of the same sort*, *and*, *similarly*, *every surface of degree n into another of that same degree*.

We shall be especially interested, later, in those surfaces, which correspond to a *sphere*. In the first place, they will be surfaces of the second degree, since a sphere is a special surface of this sort. However, since all points of the sphere are finite, so that none of them can be carried to infinity, the transformed surface must be one of the second degree which lies wholly in a finite region, i.e., it must be an *ellipsoid*.

3. Let us now see what happens to a *free vector* with the components $X = x_1 - x_2$, $Y = y_1 - y_2$, $Z = z_1 - z_2$. Using formulas (2) for the coordinates of the points 1 and 2, we get, for the components $X' = x_1' - x_2'$, $Y' = y_1' - y_2'$, $Z' = z_1' - z_2'$ of the corresponding segment $1'\,2'$,

$$\left\{ \begin{aligned} X' &= a_1 X + b_1 Y + c_1 Z \,, \\ Y' &= a_2 X + b_2 Y + c_2 Z \,, \\ Z' &= a_3 X + b_3 Y + c_3 Z \,. \end{aligned} \right. \tag{5}$$

It follows that these new components depend only upon X, Y, Z and not upon the particular values of the coordinates x_1, y_1, z_1, x_2, y_2, z_2, that is, all segments 1 2 with the same components X, Y, Z correspond to segments $1'\,2'$ with the same components X', Y', Z'. In other words, *under an affine transformation, a free vector always corresponds to another free vector*. There is essentially more in this statement than in the statement that a straight line always corresponds to a straight line. Indeed, let us take equal segments on two parallel lines, both in the same direction. Since these represent the same free vector, the corresponding segments must represent one and the same vector, i.e., they must be parallel, equal, and have the same sense (see Fig. 50).

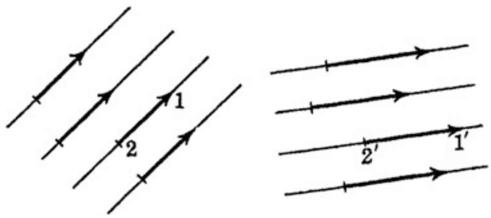

Figure 50

To every system of parallel lines there correspond again parallel lines, and to equal segments on them there correspond equal segments. These properties are rather remarkable, since – as it is easy to show – the absolute length of a segment and the absolute value of the angle between two lines are changed, in general, by an affine transformation.

4. Let us now consider *two vectors of unequal length on the same straight line.* **[78]**
One of these will be transformed into the other by multiplication by a scalar. Since X', Y', Z', in (5) are homogeneous linear functions of X, Y, Z, the corresponding vectors will differ by the same scalar factor, which means that their lengths are to each other as the lengths of the first vectors. We can state this as follows: *Two straight lines, which correspond in an affine transformation are "similar," i.e., corresponding segments on the two lines have the same ratio.*

5. Finally, let us compare two tetrahedron volumes $T = (1, 2, 3, 4)$ and $r = (1', 2', 3', 4')$. We have

$$6T' = \begin{vmatrix} x_1' & y_1' & z_1' & 1 \\ x_2' & y_2' & z_2' & 1 \\ x_3' & y_3' & z_3' & 1 \\ x_4' & y_4' & z_4' & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 + b_1y_1 + c_1z_1, & a_2x_1 + b_2y_1 + c_2z_1, & a_3x_1 + b_3y_1 + c_3z_1, & 1 \\ a_1x_2 + b_1y_2 + c_1z_2, & a_2x_2 + b_2y_2 + c_2z_2, & a_3x_2 + b_3y_2 + c_3z_2, & 1 \\ a_1x_3 + b_1y_3 + c_1z_3, & a_2x_3 + b_2y_3 + c_2z_3, & a_3x_3 + b_3y_3 + c_3z_3, & 1 \\ a_1x_4 + b_1y_4 + c_1z_4, & a_2x_4 + b_2y_4 + c_2z_4, & a_3x_4 + b_3y_4 + c_3z_4, & 1 \end{vmatrix}$$

or, applying the known theorem for multiplying determinants,

$$6T' = \begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The first factor is Δ, the second $6T$, so that we have $T' = \Delta \cdot T$. *Under affine transformations all tetrahedron volumes and hence all space volumes* (*as sums of tetrahedron volumes*, *or as limits of such sums*) *are multiplied by a constant factor*, *namely by* Δ, *the determinant of the substitution*.

These few theorems, which we have deduced from the analytic definition of affine transformation suffice to give us a *clear geometric picture* of this transformation. Their proofs have been simpler than those ordinarily given, because we had at hand, in the vector concept, the proper means for presenting them.

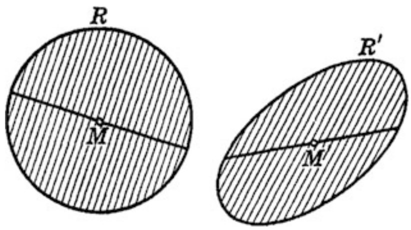

Figure 51

We get the clearest geometric picture of the affine transformation if we start with a *sphere* in the space R of the coordinates x, y, z. To this sphere, as we know, there will correspond an *ellipsoid* in the space R' of the coordinates x', y', z'. If we now consider a *system of parallel chords of the sphere*, we know, by No. 3 above, that to these chords will correspond also *parallel chords of the ellipsoid* (see Fig. 51). Further, since corresponding point rows are similar (No. 4), the *middle points of*

the chords of the sphere must also be in correspondence with the *middle points of the chords of the ellipsoid*. Since the midpoints of the chords of the sphere lie in a plane, the midpoints of the chords of the ellipsoid, by virtue of the fundamental property No. 2, must also lie in a plane, which is called a *diametral plane of the ellipsoid*. Now all diametral planes of the sphere contain its *centre* M, which bisects every chord of the sphere which passes through it (*diameter of the sphere*); hence the corresponding point M' (*centre of the ellipsoid*) lies in every diametral plane and bisects every chord through it (*diameter of the ellipsoid*). **[79]**

It is also important to see what corresponds to a system of *three mutually perpendicular diametral planes of a sphere*. This system has obviously the characteristic property that each of the three planes bisects chords parallel to the intersection of the other two planes. This property persists under affine transformation. Hence *to each of the infinitely many triples of mutually perpendicular diametral planes of a sphere there corresponds a triple of diametral planes of the ellipsoid, which have the property that chords parallel to the intersection of two of the planes are bisected by the third*. Such groups of planes are called *triples of conjugate diametral planes*; their intersections are called *triples of conjugate diameters*.

I may assume that you know that an ellipsoid contains three so-called *principal axes*, i.e., *a triple of mutually perpendicular conjugate diameters*. By what precedes, to these there must correspond under our affine transformation three mutually perpendicular diameters of the sphere in R. Let us assume, for simplicity, that the centre of the ellipsoid and the centre of the sphere are the origins in R' and R, respectively, and, by appropriate rotation, let us make these two perpendicular triples to be the x'-, y'-, z'- and x-, y-, z-axes in R' and R, respectively. It is a matter of arbitrary choice whether we think here of the space, or of the coordinate axes, as being rotated. In either case, the operation is effected by a linear homogeneous substitution of coordinates of the special sort that we have considered. Since a succession of linear homogeneous substitutions always yields another substitution of the same sort, the equations of the transformation which carries R into R' will be of the form (2) in the new coordinates:

$$\begin{aligned} x' &= a_1x + b_1y + c_1z\,, \\ y' &= a_2x + b_2y + c_2z\,, \\ z' &= a_3x + b_3y + c_3z\,, \end{aligned}$$

With the new coordinate system thus chosen, the x'-axis corresponds to the x-axis, i.e., when $y = z = 0$ so is $y' = z' = 0$. It follows that $a_2 = a_3 = 0$, and, similarly, that $b_1 = b_3 = c_1 = c_2 = 0$. *If we ignore incidental rotations, every affine transformation is a so-called "pure affine transformation"*: **[80]**

$$(6)\qquad \left\{ \begin{aligned} x' &= \lambda x\,, \\ y' &= \mu y\,, \\ z' &= \nu z\,, \end{aligned} \right. \qquad \text{where} \quad \Delta \gtrless 0\,,$$

or, as the physicists say, *a pure strain*. We may interpret these equations geometrically in the following most simple way: *space is stretched by a factor* λ (*compressed if* $|\lambda| < 1$) *parallel to the x-axis*, *and also reflected if* $\lambda < 0$; *and similarly parallel*, *with respect to the other two coordinate directions*, *by the factors* μ *and* ν *respectively*. In brief, we can look upon a pure affine transformation as a *uniform stretching of space in three mutually perpendicular directions*, which yields as clear a geometric picture as one could desire.

If we admit *oblique parallel coordinates*, the relations are still simpler. We take, in the space R, an arbitrary system of axes x, y, z, rectangular or oblique, without changing the position of the origin, and we use in R' the three straight lines, which correspond to them – due to affinity – as axes x', y', z'; these new axes will be, in general, oblique. Now the formulas for transition from rectangular to oblique coordinates, with fixed origin, are linear homogeneous equations of the form (2). Since the combination of two such substitutions always leads to another of the same sort, the equations of the affine transformation must have the form (2), even after applying the above oblique coordinates. However, with our selection of axes, they must carry the three axes of R into those of R'; hence we can conclude, after a repetition of the above argument, that the equations reduce actually to the form (6). *Thus*, *if we make use of* (*oblique*) *parallel coordinates in connection with two corresponding axis triples*, *the equations* of an affine transformation *assume the simple special form* (6).

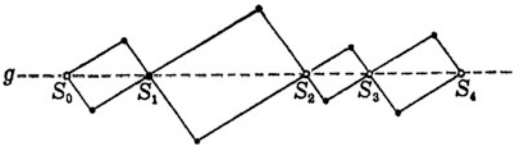

Figure 52

In connection with our discussion, there is a beautiful solution of the problem of finding a mechanism, with which one can perform affine transformations. This problem was given by me in a lecture course on mechanics, of the winter semester, 1908–09. The best solution, both with regard to the underlying conception and with regard to the convenient technical realisation of the mechanism, was furnished by
[81] Robert Remak. He used, as kinematic unit, the so-called "Nürnberg shears", i.e., a chain of jointed rods, which forms a series of similar parallelograms. The vertices common to two successive parallelograms S_0, S_1, S_2, ..., under all deformations of the jointed system, form *similar point rows* on the line g which joins them, the common diagonal of the parallelograms. (See Fig. 52.) If we fashion a triangle from three such shears by jointing them together at any of the vertices S, then the point system consisting of all the vertices S, undergoes an affine transformation with every change of the total jointed system. This will become clear (see Fig. 53) if we make an oblique coordinate system out of two of the diagonal lines of the shears. We can get additional points belonging to the same affine transformation

if we insert additional shears of the same sort between any two points S of the triangle and consider their vertices S. (In the figure, these shears are represented by their diagonal lines.) On this principle, we can set up plane and spatial models of variable affine systems of the greatest variety.[40]

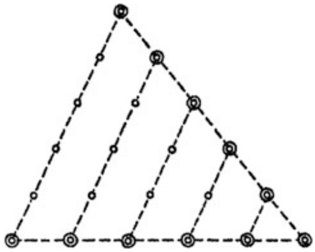

Figure 53

Application to the Theory of Ellipsoids

I shall not go farther into the discussion of properties of affine transformations. Instead, I shall show how these transformations can be used.

In the first place, an example of how they *supply an excellent device for the discovery of new geometric theorems*. The affine transformation of the sphere into the ellipsoid, explained above, enables us to get *new theorems on the ellipsoid* from known properties of the sphere. For example, if we construct three mutually perpendicular diameters of the sphere, together with the six tangent planes at their ends, we have a circumscribed cube of volume $J = 8r^3$, where r is the radius of the sphere. Our affine transformation obviously transfers each tangent plane of the sphere into a tangent plane of the ellipsoid. It follows, with the aid of the theorems above, that to the cube in space R there corresponds in space R' *a parallelepiped circumscribed about the ellipsoid*, whose faces, tangent at the ends of three conjugate diameters, are parallel to the corresponding diametral planes; and whose edges are parallel to those diameters. (Analogous relations hold in the plane for the circle and the ellipse; see Fig. 54.) The converse of this argument obviously holds also: To every parallelepiped circumscribed about an ellipsoid, in the way described above, there corresponds a cube circumscribed about the sphere, since to three conjugate diameters of the ellipsoid there correspond three mutually perpendicular diameters of the sphere. Now we know (p. [78]) that, under an affine transformation, every volume is multiplied by the determinant Δ of the substitution, so that the volume of a parallelepiped of the above sort circumscribed about an ellipsoid is given by the formula $J' = J \cdot \Delta = 8r^3 \cdot \Delta$. **[82]**

[40] A series of such models has appeared in the publishing house of Martin Schilling in Leipzig. See F. Klein and Fr. Schilling, *Modelle zur Darstellung affiner Transformationen in der Ebene und im Raume*, Zeitschrift für Mathematik und Physik, vol. 58, p. 311, 1910.

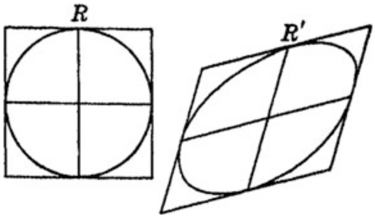

Figure 54

This formula is clearly independent of how the parallelepiped lies, so that the parallelepiped has the same constant volume, no matter to what triple of conjugate diameters it belongs. If we select, in particular, as our triple, the principal axes, which are mutually perpendicular, we get a rectangular parallelepiped whose volume is $8abc$, where $2a, 2b, 2c$ are the lengths of the principal axes. In this way we determine the constant volume, and our theorem takes the following form. *All parallelepipeds which circumscribe an ellipsoid and whose faces are parallel to three conjugate diametral planes, have the same volume* $J' = 8abc$, *where* a, b, c *are the lengths of the semi-principal axes.* In order to show that this theorem is valid for all ellipsoids, it is necessary only to see that *every* ellipsoid can be generated from a sphere by affine transformation. This follows at once from the form (6) of the equations of the affine transformation. These equations show that the axes of that ellipsoid are to each other as $\lambda : \mu : \nu$, where λ, μ, ν are three arbitrary numbers.

Although I shall confine myself to this simple example of the applications of affine transformations to theoretical geometry, I wish to emphasise even more strongly that affine transformations have the greatest significance in *practice*.

Coming first to the needs of the physicist, it is to be noted that the affine trans-
[83] formations play a fundamental role in the theory of *elasticity*, in *hydrodynamics*, and, in fact, in every branch of the *mechanics of continua*. I hardly need to explain this, for anyone who has occupied himself just once with these disciplines knows well enough that as soon as consideration is confined to sufficiently small space elements, the problem has to do with homogeneous linear deformations.

I prefer to discuss here, at greater length, the *application to correct drawing*, which is used both by the physicist and by the mathematician. *Insofar as one has to do with parallel projection, one is concerned fundamentally solely with affine transformations of space.* Unfortunately, many sins are committed in this field of correct drawing. You can find unbelievable errors in books on mathematics in the depiction of space configurations, as well as in books on physics in the representation of apparatus. To mention but one example, the sphere is very often pictured with the equator drawn as two intersecting circular arcs. (See Fig. 55.) Of course that is absurd; in fact, the correct representation is always an ellipse, as we shall see.

The principle of correct geometric drawing lies in the fact that the figure drawn is projected from a point upon the plane of the drawing. The relations are simplified if we think of that central point as lying at infinity, i.e., if we obtain the picture by means of a *family of parallel rays*. This is the case, which interests us here.

Incidentally, with these remarks we *enter the field of descriptive geometry*. I shall not give a systematic account, but I shall exhibit simply its orderly arrangement in the general edifice of geometry. Hence I shall not always give the details of proofs.

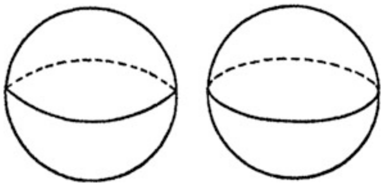

Figure 55

Parallel Projection of a Plane upon Another Plane

Let us begin by investigating the representation of a plane figure, i.e., the *projection of a plane E upon another E' by means of a family of parallel rays*. For this purpose, we choose the origin O in the intersection of E and E' (see Fig. 56), and the x-axis along this line. Choose the y-axis anywhere in E, e.g., perpendicular to the x-axis, through O, and the y'-axis as the projection of the y-axis upon E' by the parallel family, so that we have in E', in general, a system of oblique parallel **[84]** coordinates. Then the coordinates of two corresponding points of E and E' satisfy the relations

$$x' = x\,, \quad y' = \mu \cdot y\,,$$

where μ is a constant depending upon the given position of the planes and the pencil. *Thus we have actually an affine transformation*. The proof of these equations is so simple that I hardly need to state it. Moreover, these equations are specialisations of the general form (6) in that here $\lambda = 1$ and hence $x' = x$. This is due, of course, to the fact that the x-axis is the intersection of the original plane with the plane of

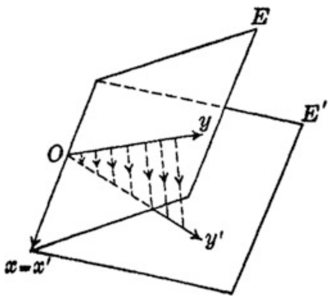

Figure 56

the drawing, so that along it each point coincides with its image. We get at once all of the essential properties of the figure if we specialise for the plane the theorems deduced earlier for space, e.g., to every circle in E there corresponds an ellipse in E', etc.

It is natural, now, to raise the *converse question*: *If two planes E and E' have a given affine relation to each other, can they be so placed that one is the parallel projection of the other?* In order to decide this, let us start from an arbitrary circle in E and the corresponding ellipse in E'. (Instead of this, we might use any two corresponding ellipses.) To the centre M of the circle there will correspond the centre M' of the ellipse (see Fig. 57). *If we now place the circle of E in the plane E' so that its centre falls at M', it will cut the ellipse in four points or not at all.* The limiting case of tangency will be disregarded, for the sake of simplicity.

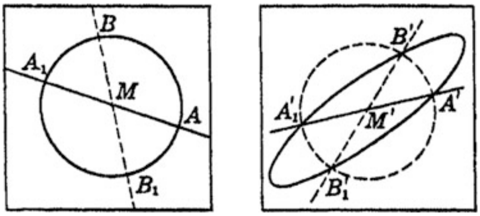

Figure 57

In the *first case*, the one shown in the figure, we consider the two diameters $A'A_1'$, $B'B_1'$, of the ellipse, which go through the four points of intersection lying in E'. Corresponding to these – and equal to them by construction – are two diameters AA_1, BB_1, of the circle in E. Hence, by reason of a general property of affine transformations (No. 4, p. [78]), corresponding segments on AA_1 and $A'A_1'$, as well as on BB_1 and $B'B_1'$, are equal. Now, if we lay the plane E upon E' so that M falls upon M' and so that one of these pairs of straight lines, say AA_1 and $A'A_1'$,
[85] coincides, and if we then rotate E about this line as axis up into space, we have an affine transformation of the two planes, under which each point of their line of intersection corresponds to itself. Then it is easy to show, though I shall not carry out the proof, that, no matter what the angle between the planes may be, the joins of corresponding points are all parallel to each other, i.e., that the affine transformation between the two planes can, in fact, be effected by parallel projection.

If, however, our circle does not cut the ellipse, i.e., if its radius is smaller than the small semi-axis of the ellipse or larger than the large one, then, in the language of analysis, the two common diameters are imaginary and are not available for use in drawing; hence the construction is impossible. If it is still desired to bring about parallel projection, it becomes necessary to employ a *similarity transformation*, and to expand or shrink the circle by that transformation until the first case appears. We use such similarity transformations constantly in the making of pictures, in order to "change the scale." Thus we reach finally the *main theorem*, *that any*

affine relation between two planes can be effected in infinitely many different ways through combination of a similarity transformation with a parallel projection.

Axonometric Mapping of Space (Affinity with Vanishing Determinant)

We go over now to the *problem of representing all of space upon a plane by means of parallel projection*, which is much more important and interesting than this mapping of one plane upon another. To avoid tedious details, we shall agree always to admit a stretching or a shrinking of the picture by means of a similarity transformation. There arises, thus, the process, which is called *axonometry* in descriptive geometry. This process plays an extraordinarily important role in practice. Every photograph is very nearly an axonometric mapping, if the object is only far enough away from the camera. (Strictly speaking, it is a central projection.) Exact axonometry is used especially, however, in most of the cases in which we wish to map geometric figures in space, physical apparatus, architectural parts, and so on. Very interesting examples of all sorts of axonometric mappings, which are also directly useful in teaching, can be found in the book entitled *Leitfaden der Projectionslehre* by Conrad H. Müller and O. Pressler.[41] It is shown there, for example, how to draw accurately a tangent compass, a drum armature, crystals of the most varied kinds, and, to cite examples from the entirely different field of biology, cellular tissue, a beehive, and many other things.

Let me now state the theorem which connects axonometry with our discussion of affine transformations: *The mapping of space upon a plane by means of parallel projection and similarity transformation* (*axonometry*) *is effected analytically by an affine transformation* w i t h a v a n i s h i n g d e t e r m i n a n t: **[86]**

$$(1) \qquad \begin{cases} x' = a_1x + b_1y + c_1z \\ y' = a_2x + b_2y + c_2z\,, \\ z' = a_3x + b_3y + c_3z \end{cases} \qquad \text{where} \quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0\,.$$

This is precisely the exceptional case, which we postponed. Thus you see the importance of these "degenerate" transformations, which unfortunately are often unduly neglected. The converse is also true, namely, that *every such substitution*, *with* $\Delta = 0$, *gives an axonometric mapping*. This presupposes, to be sure, that neither all the coefficients of the substitution nor all the minors of second order vanish, for these possibilities would imply still further degenerations, which I shall pass over, since they can be investigated readily according to the following plan.

In order to prove our assertion, let us convince ourselves *that all points* x', y', z' *given by* (1) (*for arbitrary* x, y, z) *actually lie in one plane*, i.e., that there are three

[41] Ein Übungsbuch der konstruierenden Geometrie, Leipzig, 1903.

members k_1, k_2, k_3, such that we have

$$(2) \qquad k_1 x' + k_2 y' + k_3 z' = 0$$

identically in x, y, z. By (1), this identity is equivalent to the three homogeneous linear equations

$$(2') \qquad \begin{cases} k_1 a_1 + k_2 a_2 + k_3 a_3 = 0\,, \\ k_1 b_1 + k_2 b_2 + k_3 b_3 = 0\,, \\ k_1 c_1 + k_2 c_2 + k_3 c_3 = 0\,, \end{cases}$$

and these determine precisely the ratios $k_1 : k_2 : k_3$ uniquely, provided that the determinant Δ vanishes but that the nine minors are not all zero. Hence all the image points x', y', z' actually lie in the plane (2) determined by the equations (2′).

We shall now introduce, in the space R', a new rectangular coordinate system such that the plane (2) becomes the x'-y'-plane ($z' = 0$). Then there must correspond to every point of R a point in $z' = 0$, and the equations of our affine transformation, in the new coordinates, will have necessarily the form

$$(3) \qquad \begin{cases} x' = A_1 x + B_1 y + C_1 z\,, \\ y' = A_2 x + B_2 y + C_2 z\,, \\ z' = 0. \end{cases}$$

[87] The *six constants* $A_1, \ldots, C_2$ *are completely arbitrary*, since the determinant of the substitution vanishes in any case, by reason of the special form of the last row. The three minors may not all vanish however; that is

$$A_1 : B_1 : C_1 \neq A_2 : B_2 : C_2\,;$$

otherwise we should have the degeneration that we excluded above.

I shall now give the proof that the mappings of the space R upon the x'-y'-plane E', defined analytically as above, are identical with the axonometric projections defined above. I shall present the proof in separate steps, by developing the chief properties of the transformation (3), much as we discussed earlier (pp. [75] sqq.) the affine transformations with non-vanishing determinant.

1. In the first place, it is clear that to every point x, y, z of R there corresponds a unique point (x', y') in E'. Conversely, given a point (x', y') in E', the equations (3) show that the corresponding point (x, y, z) in R lies in two definite planes whose coefficients, by our assumption, are not proportional, and which have, therefore, a line of intersection, lying in the finite domain. All the points of this straight line must correspond, in our transformation, to the same point (x', y'). If we now allow the point (x', y') to vary, each of the two planes will be moved parallel to itself, since the coefficients A_1, B_1, C_1 and A_2, B_2, C_2 remain unchanged. Thus their line of intersection remains parallel to itself, and we have the result *that to each point of E' there correspond all the points of one line of a double infinity of parallel lines in R*. This indicates immediately the connection between our mapping and the parallel projection of space.

2. Just as in No. 3 (p. [77]) under the general affine transformation, we find now for the components X', Y' of the segment in E' which corresponds to the free vector X, Y, Z of R, the formulas

$$\begin{cases} X' = A_1 X + B_1 Y + C_1 Z\,, \\ Y' = A_2 X + B_2 Y + C_2 Z\,, \\ Z' = 0\,. \end{cases} \tag{4}$$

These show again that *to every free vector in R there corresponds a free vector X', Y' of the picture plane E'*, or, more precisely, if one displaces a segment in space R parallel to itself, preserving its length and direction, the corresponding segment in the plane E' also moves parallel to itself and maintains its length and direction.

3. We consider in particular the *unit vector* $X = 1$, $Y = Z = 0$, on the x-axis, which goes from $(0, 0, 0)$ to $(1, 0, 0)$. To it there corresponds in E', by (4), the vector $X' = A_1$, $Y' = A_2$, which goes from the origin O' to the point whose coordinates are (A_1, A_2). In precisely the same way, there correspond to the unit vectors on the y- and the z-axes the two vectors from O' to the points (B_1, B_2) and **[88]** (C_1, C_2), respectively. These three vectors in E, which we shall call for brevity, (A), (B), (C) (see Fig. 58), can be chosen arbitrarily, since the coordinates of their endpoints determine the six arbitrary parameters of the affine transformation (3), so that they completely determine the mapping. Yet, these three vectors *must not all lie in the same line*, and we shall assume, for simplicity, that no two of them lie in one line. The result is as follows: *The three unit vectors on the coordinate axes of R are mapped upon three arbitrary vectors through the origin O' in E', which, when they are known, completely determine the affine transformation.*

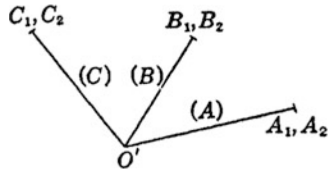

Figure 58

4. In order to obtain *geometrically* the mapping of (A), (B), (C), we start from any point p $(x, y, z = 0)$ of the x-y-plane. We obtain the vector from O to p by multiplying the unit vector of the x-axis by the scalar number x, and that of the y-axis by the number y, and by then adding the product vectors (see Fig. 59).

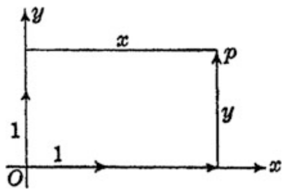

Figure 59

However, we can transfer this construction at once to E', since the relation between the x-y-plane and E' is obviously an ordinary two-dimensional affine transformation (with non-vanishing determinant). *We obtain, then, the image point p' of p by means of the scalar multiplication of the vectors* (4) *and* (B) *by x and y, respectively, and the addition of the products by the parallelogram law* (Fig. 60). In this way, we can construct in E', the map of any point, and hence, point by point, any figure of the x-y-plane.

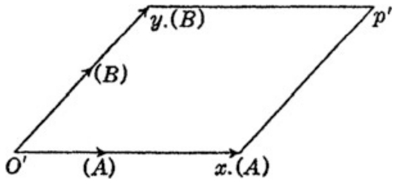

Figure 60

5. If we transfer these considerations to an arbitrary point of the space R, we can prove easily (see Fig. 61) the following result: *We obtain the image point p' of a point p whose coordinates are* (x, y, z), *if we apply the parallelogram law for addition to the products of the vectors* (A), (B), (C) *by x, y, and z, respectively.* Since addition is commutative, we can perform this construction in $1 \cdot 2 \cdot 3 = 6$ different ways, and we get p' as the terminal point of six different polygonal paths, consisting of additive combinations of parallel and equal segments. The figure thus constructed (see Fig. 61) is obviously the representation of that rectangular paral-
[89] lelepiped in the space R, which is bounded by the three coordinate planes and the planes through p parallel to them. We are accustomed, from our youth on, to look upon such plane figures as pictures of solid figures, especially when the appearance is increased by drawing the front edges in heavier lines. This habit is so strong that this mapping of the parallelepipedon seems almost trivial, whereas it represents really a very noteworthy theorem.

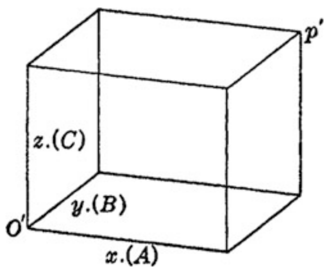

Figure 61

6. With the aid of this last construction, we can make in E' the picture of any figure in space, i.e., of all of its points. I shall consider only one example: If we have

a *sphere*, with radius 1 and centre at the origin O, then we shall consider primarily the circles in which it cuts the coordinate planes. The circle of intersection in the x-y-plane has the unit vectors on the x- and the y-axes as conjugate, i.e., as mutually perpendicular semi-diameters. Since we have an affine relation, there will correspond to it an ellipse (see Fig. 62) which has O' as centre and the vectors (A) and (B) as conjugate semi-diameters, and which is thus inscribed in the parallelogram formed by the vectors $2(A)$ and $2(B)$. In the same way, the ellipses corresponding to the other two circles of intersection will have O' as centre and (B), (C) and (A), (C) as conjugate semi-diameters.

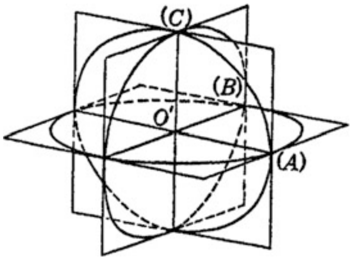

Figure 62

The Fundamental Theorem of Pohlke

7. Now that we have obtained a complete idea, showing the nature of the affine transformations (3) with vanishing determinant, we must take the last, decisive, step in our considerations, and show, namely, that these affine transformations actually arise through axonometric projection, as we have asserted. This requires, chiefly, the so-called *fundamental theorem of Pohlke*, which Karl Wilhelm Pohlke, professor of descriptive geometry at the School of Architecture (*Bauakademie*) in Berlin, discovered in 1853 and published in his *Lehrbuch der darstellenden Geometrie*[42] in 1860. Hermann Amandus Schwarz published in 1863[43] the first elementary proof of this theorem and gave, at the same time, a sketch of the interesting history of its discovery, which you should read.

Pohlke himself did not define axonometry analytically, but geometrically, as **[90]** a mapping of space by means of parallel rays (together, where necessary, with a similarity transformation). His theorem stated *that the three unit vectors on the coordinate axes of space could go over*, *under such a mapping*, *into three arbitrary vectors of E' through O'*. That our analytically defined mapping actually led to three such vectors was apparent in No. 3; hence for us the underlying significance

[42] Two parts, 4th edition, Berlin, 1876. This theorem is in Part I, p. 109.

[43] Elementarer Beweis des Pohlkeschen Fundamentalsatzes der Axonometrie. *Journal für die reine und angewandte Mathematik*, vol. 63, pp. 309–314 = *Gesammelte Mathematische Abhandlungen*, vol. 2, p. 1, Berlin, 1890.

of Pohlke's theorem is *that our analytically defined mapping* (3) (*p.* [80]) *is effected by parallel projection and change of scale*, whereby the parallel lines mentioned in No. 1 become projecting rays.

8. I should like to indicate an approximate plan for a direct analytical proof of the theorem thus formulated. If we fix our attention upon the two families of parallel planes in R:

$$A_1x + B_1y + C_1z = \xi\,, \quad A_2x + B_2y + C_2z = \eta\,,$$

where ξ and η are variable parameters, then each pair of values of ξ and η determines one of the parallel lines in question. Now, if it were possible to place in the space R a picture plane E' containing a rectangular coordinate system x', y' with an appropriate unit of length, so that each ray ξ, η would pierce the plane E' in the point $x' = \xi$, $y' = \eta$, then the mapping (3) would actually be brought about geometrically, as desired.

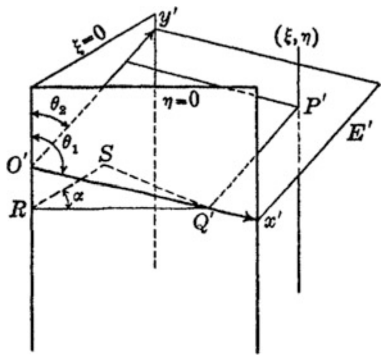

Figure 63

To this end, the planes $\xi = 0$, $\eta = 0$ must cut the plane E' in the coordinate axes $O'y'$ and $O'x'$ respectively, i.e., in mutually perpendicular lines. If θ_1, θ_2 (determining the position of E') are the angles between these axes and the line $\xi = \eta = 0$ (see Fig. 63), and if we denote by α the (known) angle between the planes $\xi = 0$ and $\eta = 0$, then, applying the cosine theorem of spherical trigonometry to the trihedral angle formed by $\xi = 0$, $\eta = 0$, and E', we find the cosine of the angle of $O'x'$, $O'y'$ to be

$$\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\alpha\,,$$

and this angle is a right angle if

(a) $$\cot\theta_1 \cdot \cot\theta_2 = -\cos\alpha\,.$$

Now every plane $A_1x+B_1y+C_1z = \xi$ cuts E' in a straight line $x' =$ constant. If
[91] Q' is its intersection with the x'-axis, then the corresponding x'-value, to within the undetermined scale factor λ of the coordinate system in E', is equal to $O'Q'$. If we

drop perpendiculars $Q'S$ and $Q'R$ upon the plane $\xi = 0$ and the line $\xi = \eta = 0$, respectively, we shall have

$$O'Q' = \frac{Q'R}{\sin\theta_1}, \quad Q'R = \frac{Q'S}{\sin\alpha}$$

and since $Q'S$, as the common perpendicular between the planes

$$A_1x + B_1y + C_1z = 0 \quad \text{and} \quad A_1x + B_1y + C_1z = \xi$$

is easily expressed by means of a known formula of analytic geometry of space, it follows finally that

$$x' = \lambda \cdot O'Q' = \lambda \frac{\xi}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sin\theta_1 \cdot \sin\alpha}.$$

Similarly, we find as the y' coordinate of the points of intersection of $A_2x + B_2y + C_2z = \eta$ and E',

$$y' = \lambda \cdot \frac{\eta}{\sqrt{A_2^2 + B_2^2 + C_2^2} \cdot \sin\theta_2 \cdot \sin\alpha}.$$

Now, since we wish each parallel ray determined by the parameter values ξ, η to pierce the plane E' in the point $x' = \xi$, $y' = \eta$, we must have:

$$\text{(b)} \qquad \lambda = \sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sin\theta_1 \cdot \sin\alpha = \sqrt{A_2^2 + B_2^2 + C_2^2} \cdot \sin\theta_2 \cdot \sin\alpha,$$

from which we get the second equation for θ_1, θ_2:

$$\text{(c)} \qquad \sin\theta_1 \cdot \sqrt{A_1^2 + B_1^2 + C_1^2} = \sin\theta_2 \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}.$$

A very simple calculation shows that the equations (a) and (c) have only one real pair of solutions for $\cot\theta_1$ and $\cot\theta_2$, determined except for the sign; i.e., *there is essentially only one position* (*of course symmetric to the common normal plane of* $\xi = 0$, $\eta = 0$) *of the plane* E', *in which the affine transformation* $x' = \xi$, $y' = \eta$ *is axonometrically realised*, insofar as we choose the scale of the rectangular coordinate system in E according to (b). We can give this whole argument a more geometric form if we start from the condition that the unit points of the x'- and y'-axes fall upon the straight lines $\xi = 1$, $\eta = 0$ and $\xi = 0$, $\eta = 1$. Then the problem is to find a plane E' which cuts a given triangular prism in an isosceles right triangle.

After this detailed presentation, I hardly need to discuss further the converse theorem, already mentioned, *that every axonometric projection represents an affine transformation with a vanishing determinant.* This converse can be verified by using first, as we did earlier (p. [83]), the oblique coordinate system in the plane of

projection E' which arises by parallel projection from the x- and y-axes in R and then, by means of a linear substitution, going over to the initially given rectangular coordinate system in E'.

[92] In closing this chapter on affine transformations, I should like to remind you that we can get an illustration of axonometric representation experimentally by using a projection lamp (one must think of it as infinitely remote) *to throw shadow pictures of simple models* (square, circle, ellipse, cube) upon a projection screen. We should get, in this way, a confirmation of our results and our figures; and, in particular, we could easily check experimentally the theorem of Pohlke, by subjecting the shadow picture of three mutually perpendicular rods to all sorts of change by movements of the model as well as of the screen.

We go over, now, to a new chapter, which treats of more general transformations, including affine transformations as special cases, namely, the *projective transformations*.

II. Projective Transformations

In this chapter also, I should like to deal with space of three dimensions from the first.

Analytic Definition; Introduction of Homogeneous Coordinates

1. I shall begin with the *analytic definition of the projective transformation*. We now take x', y', z', no longer as integer, but as *fractional linear functions* of x, y, z, but with the condition, which is essential, that they all have the same denominator:

$$(1)\qquad \begin{cases} x' = \dfrac{a_1x + b_1y + c_1z + d_1}{a_4x + b_4y + c_4z + d_4}\,, \\ y' = \dfrac{a_2x + b_2y + c_2z + d_2}{a_4x + b_4y + c_4z + d_4}\,, \\ z' = \dfrac{a_3x + b_3y + c_3z + d_3}{a_4x + b_4y + c_4z + d_4}\,. \end{cases}$$

To every point x, y, z there corresponds, accordingly, a definite finite point x', y', z', provided only that the common denominator is not zero. If, however, the point x, y, z approaches the plane $a_4x+b_4y+c_4z+d_4 = 0$, the corresponding point x', y', z' – this is the novelty, as compared with the affine transformation – moves to infinity: it "vanishes," in a sense. We call that plane, therefore, the *vanishing plane*, its points *vanishing points*, and we say that it corresponds, in the projective transformation, to the part of space at infinity, or to the points at infinity.

2. In the treatment of the problems arising here, it is very convenient, as you know, to use *homogeneous coordinates*, i.e., in place of the three point coordinates x, y, z, to use four quantities ξ, η, ζ, τ, defined by the equations

$$x = \frac{\xi}{\tau}\,, \quad y = \frac{\eta}{\tau}\,, \quad z = \frac{\zeta}{\tau}\,.$$

These four quantities are to *vary independently of each other*, *but not all four are to vanish simultaneously*, and *none of them is to become infinite*. To every **[93]**

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_8

point x, y, z there will then belong infinitely many systems of values $\rho\xi$, $\rho\eta$, $\rho\zeta$, $\rho\tau$, where ρ is an arbitrary factor ($\neq 0$). Conversely, every system of values ξ, η, ζ, τ where $\tau \neq 0$, determines a definite finite point x, y, z (all systems $\rho \cdot \xi$, $\rho \cdot \eta$, $\rho \cdot \zeta$, $\rho \cdot \tau$ give the same point). When $\tau = 0$, one, at least, of the quotients x, y, z becomes infinite, and we stipulate, accordingly, that *every system of values* ξ, η, ζ, $\tau = 0$ *shall signify an "infinitely distant point,"* and, indeed, all systems $\rho\xi$, $\rho\eta$, $\rho\zeta$, 0 represent one and the same point. In this precise analytic way we introduce the points, which, as "infinitely distant," are added to the ordinary finite points.

Experience shows that operation with homogeneous coordinates produces, at least with beginners, something like physical discomfort. I believe that the somewhat indefinite, fluid quality of these quantities, which the arbitrary factor ρ brings in, is the cause of this feeling, and I hope that such a statement may help to allay this discomfort.

With the same end in view, some incidental remarks may be helpful about *certain geometric representations,* which can be associated with homogeneous coordinates. I shall speak first only of a *plane E*. In this case, let us write for the two rectangular coordinates

$$x = \frac{\xi}{\tau}, \quad y = \frac{\eta}{\tau}.$$

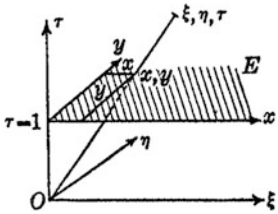

Figure 64

We now interpret ξ, η, τ as *rectangular coordinates in space* and, in this space, we choose the plane $\tau = 1$, parallel to the ξ-η-plane, as the plane E (see Fig. 64). In this plane E, put $x = \xi$, $y = \eta$. If we now join the point x, y of E to O by a straight line, then, for points on this line, ξ/τ and η/τ are constant and we may write

$$\frac{\xi}{\tau} = x, \quad \frac{\eta}{\tau} = y,$$

since, for $\tau = 1$, we should have $\xi = x$, $\eta = y$. *Accordingly, the introduction of homogeneous coordinates signifies simply the mapping of the plane E into that family of rays projecting from the origin O of the three-dimensional auxiliary space, of which E is a section. The homogeneous coordinates of a point are the space coordinates of the points of the projecting family of rays of that point.* Since to each point of E there correspond the infinitely many points of the ray, the significance of
[94] the indefiniteness of the homogeneous coordinates is made clear. The exclusion of the system of values $\xi = \eta = \tau = 0$ has its geometric basis in the fact that the point

O alone determines no ray, and hence no point in E. Moreover, it is obvious that we need no infinite values of ξ, η, τ, since we get all rays by joining O with finite points. Finally, it is clear that we avoid infinitely large values of the coordinates by replacing the infinite region of the plane E by the parallel rays through O given by $\tau = 0$.

Moreover, the common expression "the line at infinity" finds here its intuitive geometric meaning. Analytically, it is only the expression of the abstract analogy that all "infinitely distant points" satisfy the linear equation $\tau = 0$, just as every finite straight line has a linear equation. But now we can say geometrically that to every line of E there belongs a plane family of the space family O; and, conversely, every plane family in the space family O determines a straight line in E, except the plane family $\tau = 0$. Hence it seems appropriate to designate as a straight line the set of points in E that correspond to this family $\tau = 0$, and so we have "the infinitely distant straight line."

We can form similar representations if we introduce homogeneous coordinates into *space of three dimensions*. We think of the space as a section $\tau = 1$ of a four-dimensional auxiliary space ξ, η, ζ, τ, and we relate it to the space family, which projects it from the origin in the auxiliary space. We can then carry through without difficulty all the other considerations in almost word-for-word analogy with what preceded, and, in particular, we can transfer the interpretation of the infinitely distant elements. In this, the use of four-dimensional space is only a convenient means of expression, to which no mystical significance is to be attached.

3. If we now introduce into the equations (1) of the projective transformation *homogeneous coordinates for both* spaces R and R', we can separate them, by introducing an arbitrary proportionality factor ρ', since they all have the same denominator, into the following four equations:

$$(2) \qquad \begin{cases} \rho'\xi' = a_1\xi + b_1\eta + c_1\zeta + d_1\tau\,, \\ \rho'\eta' = a_2\xi + b_2\eta + c_2\zeta + d_2\tau\,, \\ \rho'\zeta' = a_3\xi + b_3\eta + c_3\zeta + d_3\tau\,, \\ \rho'\tau' = a_4\xi + b_4\eta + c_4\zeta + d_4\tau\,. \end{cases}$$

Leaving out of account the arbitrary factor ρ', we see that this is the most general homogeneous linear transformation in four variables; hence it represents an *affine relation of the two four-dimensional auxiliary spaces* P_4, P'_4, in which we can interpret the homogeneous coordinates in the manner explained in No. 2. All this can be **[95]** represented more concretely if we again limit ourselves to the *plane*. *We obtain the most general projective transformation of a plane if we apply an arbitrary affine transformation to the space of that space family*, *with fixed centre O*, *which projects this plane*, *and then cut the plane with the transformed family*. We always get, in this way, the same projectivity of our space, corresponding to the factor ρ', if we add a similarity transformation from O. For this transforms into itself each of the rays through O, and the projectivity depends solely upon the intersections of these with the plane.

The procedure which we have followed, in using the auxiliary spaces P, P', is called the *principle of projection and section*. It is often very useful in that, generally speaking, it *makes complicated relations in space of n dimensions appear simpler and easier to understand*, *through auxiliary considerations in spaces of* $n + 1$ *dimensions*.

4. We shall now reverse the transformation equations (2) for ξ, η, ζ, τ. The theory of determinants shows *that* ξ, η, ζ, τ *are likewise linear homogeneous combinations of* ξ', η', ζ', τ', of course with a proportionality factor ρ;

$$\text{(3)} \qquad \begin{cases} \rho\xi = a_1'\xi' + b_1'\eta' + c_1'\zeta' + d_1'\tau' \,, \\ \rho\eta = a_2'\xi' + b_2'\eta' + c_2'\zeta' + d_2'\tau' \,, \\ \rho\zeta = a_3'\xi' + b_3'\eta' + c_3'\zeta' + d_3'\tau' \,, \\ \rho\tau = a_4'\xi' + b_4'\eta' + c_4'\zeta' + d_4'\tau' \,. \end{cases}$$

provided only that the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

of (2) *does not vanish. The systems of values* ξ, η, ζ, τ *and* ξ', η', ζ', τ' *are thus in one-to-one correspondence* (*to within those arbitrary common factors*).

Let me say, however, as you might expect after our experience with the affine transformations, that the case $\Delta = 0$ is here also especially interesting, and that it may not be disregarded. It represents the mapping of space upon a plane, as in every central projection, e.g., in photography. For the present, however, we shall consider the general case $\Delta \neq 0$.

Geometric Definition: Each Collineation is a Projectivity

5. It follows at once from (2) and (3) that, when a linear relation exists between ξ, η, ζ, τ, there is also one between ξ', η', ζ', τ', and conversely. *To every plane*
[96] *there corresponds a plane*; in particular, to the infinitely distant plane of R' there corresponds a definite and, in general, a finite plane in R, i.e., the *vanishing plane* mentioned above. Thus the terminology of the plane at infinity proves to be highly convenient, since only it permits the statement of such theorems as valid without exception. It follows, further, that *to every straight line there corresponds necessarily a straight line*. In the terminology of Möbius (p. [76]), *every projective transformation is a collineation*.

6. Now it is the beautiful that the converse is also true: *Every collineation of space*, *i.e.*, *every reversibly unique transformation such that to every straight line*

there corresponds a straight line, and which satisfies certain other almost self-evident conditions, is a projectivity, i.e., a transformation defined analytically by equations (1) or (2).

For the sake of convenience, I shall give here Möbius' proof only for the plane; for space we should proceed similarly. The *plan of the proof* is as follows. From an arbitrary collineation, we select two corresponding point quadruples and we shall show (a) that there is always a projectivity, which transforms two such quadruples into each other. However, a projectivity is also a collineation; and we shall prove (b) that, under certain conditions, there can be only *one* collineation in which these quadruples can correspond to each other. Thus the projectivity must, in fact, be identical with the given collineation, which proves the theorem. We shall now give the *details* of these two steps.

(a) We remark that the equations of the projectivity in the plane:

$$\begin{aligned}
\rho'\xi' &= a_1\xi + b_1\eta + d_1\tau\ ,\\
\rho'\eta' &= a_2\xi + b_2\eta + d_2\tau\ ,\\
\rho'\tau' &= a_3\xi + b_3\eta + d_3\tau
\end{aligned}$$

contain $9 - 1 = 8$ constants. (A change in ρ' does not alter the transformation.) That two given points may correspond to each other in a projectivity requires two linear conditions for the constants of the projectivity, since we are concerned only with the *ratios* of the three homogeneous coordinates. The correspondence of two point quadruples represents thus $2 \cdot 4 = 8$ linear conditions, or, more precisely, eight linear homogeneous equations for the nine quantities $a_1, \ldots, d_3$. Such equations always have a solution, as you know; hence *we have found in this manner the constants of a projectivity, which transforms the given quadruples into each other*. We can guarantee, to be sure, that this is a *proper* projectivity with a *non-vanishing* determinant, and that it is uniquely determined, only if each of the given point quadruples is "in general position," i.e., if no three points of a quadruple are collinear; but it is only for this case that we need the theorem.

(b) We now think of an arbitrary collineation of the planes E and E'. If, then, 1, **[97]**
2, 3, 4 are any four points of E, of which no three are collinear, and if $1', 2', 3', 4'$ are the corresponding points in E', satisfying the same condition, then our assertion is *that the collineation is completely determined by the correspondence between these two quadruples of points*. We shall give this proof by showing that this collineation can be built up in one and only one way from these two corresponding quadruples by using solely their two characteristics properties (uniqueness, and the mutual correspondence of straight lines). As our chief aid, we shall use the so-called *Möbius' net*, that is systems of straight lines, which we spread over the plane after the manner of a spider's web.

To begin with, we draw, in each plane (see Fig. 65) the six lines joining the four points by pairs. These must correspond in the collineation, for, to the straight line 1 2 there must correspond a straight line in E which must contain $1'$ as the image of 1, as well as $2'$, the image of 2; and that could be only the line $1'\,2'$. Similarly, the points arising as intersections of corresponding straight lines must themselves

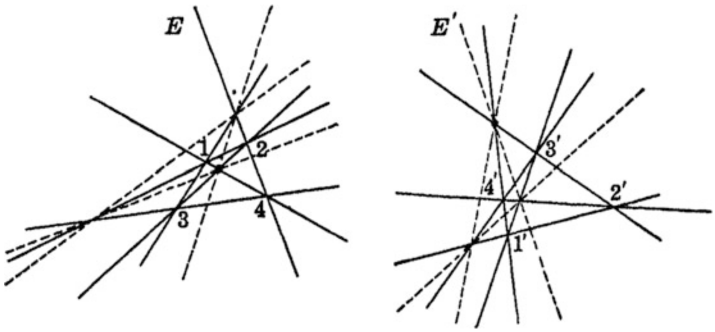

Figure 65

correspond, e.g., the points (1 4, 2 3) and $(1'\,4', 2'\,3')$: this follows immediately from the collinearity and the uniqueness. If we join the new points by straight lines, extend these to intersection with the earlier lines, join the resulting points of intersection again, and continue this process, there will appear in *each plane a net of points and straight lines which gets denser and denser*, *and these points and straight lines must correspond in pairs in the desired collineation*.

If we now select an arbitrary point in E, say, either it will be itself one of the vertices of the net, or else it is easy to show that we can enclose it in meshes of
[98] the net, which can become indefinitely tighter, i.e. as limit point of vertices. In the first case the corresponding point in E' is uniquely determined as the corresponding net vertex. In order to take care of the second case, we must make an *addition to the definition* of collineation, one which to Möbius seemed so self-evident that he did not think it required an explicit statement. It is, namely, *that the mapping shall be continuous*, *i.e.*, *each limit point of a point set in E shall be in correspondence with the limit point of the corresponding point set in E'*. From this, and from the preceding remarks, it follows also in the second case that the corresponding point in E' is uniquely determined. We have established then the *correctness of our assertion* 6, *insofar as the collineation is continuous*. In the same way we could prove that a continuous collineation in ordinary space is determined by five pairs, and in space of n dimensions by $n + 2$ pairs, of corresponding points.

Returning to the considerations at the beginning of No. 6 (p. [96]), we have, as one result, the following precise theorem. *Projective transformations are the only continuous reversibly unique transformations, which always carry lines into lines*.

After this digression, let us resume the investigation begun in No. 5 (pp. [95]–[96]) of *the behaviour of the fundamental geometric manifolds* under projective transformations, or, as we can now say, under *collinear transformation*. We saw there that an unlimited plane or straight line is carried over by projection into a figure of the same sort, so that these concepts have a definite invariable significance with respect to projectivities. In this property, the general projectivities agree with the affine transformations. They differ from them however in their behaviour with respect to parallelism.

Behaviour of Basic Configurations Under Projectivities

7. *Behaviour with respect to the concept of parallelism.* Indeed, the parallelism of two straight lines is not necessarily maintained under projective transformations, as it was under affine transformation (p. [77]). On the contrary, the infinitely distant plane of one space can go over into any finite plane whatever (the vanishing plane), of the other, and there will correspond, thereby, in general, to the point at infinity common to two parallels, a finite point of the vanishing plane in which the two straight lines intersect that correspond to the parallels. By the aid of homogeneous coordinates we can follow this exactly. To be sure, we see here, also, that the concept of parallelism is not ruthlessly disturbed, but that it becomes a part of a perfectly definite general concept. *The infinitely distant points of space constitute a plane, which can be carried over by projection into any other (finite) plane of space, and which, to this extent, has equal status with all these planes.* It is characterised as *arbitrary*, only to a certain degree, by the descriptive phrase "*the infinitely distant.*" *Straight lines (and planes also) are then called parallel if their* **[99]** *intersection lies on this special plane.* By a projective transformation they may be carried into lines (or planes), which meet on *another* fixed plane, in which case the new straight lines (or planes) are said to be no longer parallel.

With this property there is connected the fact *that the fundamental configurations of Graßmann, likewise, have no invariant significance under projection.* The free vector is by no means carried over into another free vector, the line-bound vector into another such, etc. In fact, let us look at a line segment of space R, with the six coordinates.

$$X = x_1 - x_2\,, \qquad Y = y_1 - y_2\,, \qquad Z = z_1 - z_2\,,$$
$$L = y_1 z_2 - y_2 z_1\,, \quad M = x_2 z_1 - z_2 x_1\,, \quad N = x_1 y_2 - y_1 x_2$$

and let us set up the analogous quantities X', ..., N' out of the coordinates of the points $(x'_1,\ y'_1)$ and $(x'_2,\ y'_2)$ which correspond to (x_1, y_1) and (x_2, y_2) under the projective transformation (1) (p. [92]):

$$x'_1 = \frac{a_1 x_1 + b_1 y_1 + c_1 z_1 + d_1}{a_4 x_1 + b_4 y_1 + c_4 z_1 + d_4} \text{ etc.,} \qquad x'_2 = \frac{a_1 x_2 + b_1 y_2 + c_1 z_2 + d_1}{a_4 x_2 + b_4 y_2 + c_4 z_2 + d_4} \text{ etc.}$$

By these formulas, X', ..., N' become fractions whose numerators, to be sure, appear as linear combinations solely of the six quantities X, ..., N, with constant coefficients, while the denominator common to all of them,

$$(a_4 x_1 + b_4 y_1 + c_4 z_1 + d_4)(a_4 x_2 + b_4 y_2 + c_4 z_2 + d_4),$$

contains the point coordinates themselves and cannot be expressed in terms of X, ..., N alone. Thus the coordinates of the transformed line segment depend not only on those of the original segment but also on the special position of its end points. If we slide the segment (1 2) along its line, so that X, ..., N do not change, X', ..., N' will change, in general, i.e., *the segment* $(1'\,2')$ *is not a line segment in the Graßmann sense.* That the *unlimited straight line* persists as such, nevertheless,

under projective transformation, follows from the fact that it is represented by the *ratios* of the quantities $X' : Y' : \ldots : N'$, from which the disturbing common denominator disappears by cancellation. Thus these ratios are actually expressed solely in terms of the ratios $X : Y : \ldots : N$.

8. There remain still some important *configurations, which go over into configurations of the same sort under projective transformation*. In the first place, every quadratic equation in x', y', z' arises from a quadratic equation in x, y, z, as we see by multiplying through by the square of the common denominator $a_4x+b_4y+c_4z+d_4$, and conversely. This shows *that every surface of the second degree in a space R*
[100] *corresponds to one of the same nature in R'*. Therefore every intersection of such a surface with a plane, i.e., every *curve of order two in a space R corresponds to one of the same nature in R'*. *In the same way*, *any algebraic configuration*, *defined by one or several equations in the coordinates*, *will be transformed into a configuration of the same sort*; the nature of these configurations is thus invariant under projective change.

9. Along with these invariant confiugrations, defined by equations, I must mention a *numerical quantity* whose value remains unchanged under all projective transformations. It offers a certain substitute for the concepts *distance* and *angle*, whose values, as you know, are not invariant even under affine transformations, to say nothing of projective transformations. Speaking first of the *straight line*, let us consider a *certain function of the distances among four arbitrarily selected points* 1, 2, 3, 4, namely, the *cross-ratio* mentioned on p. [6]:

$$\frac{\overline{12}}{\overline{14}} : \frac{\overline{32}}{\overline{34}} = \frac{\overline{12}\cdot\overline{34}}{\overline{14}\cdot\overline{32}}\,.$$

In fact we can easily verify (by calculation), the invariance of this quantity under projective transformation, and we shall actually do so later in another connection. (See pp. [157]–[158])

The case is quite similar for *families of rays*, except that we use, not the angles themselves, but their *sines*. Thus, if 1, 2, 3, 4 are rays or planes of a family, their cross-ratio is the expression

$$\frac{\sin(1,2)}{\sin(1,4)} : \frac{\sin(3,2)}{\sin(3,4)} = \frac{\sin(1,2)\sin(3,4)}{\sin(1,4)\sin(3,2)}\,.$$

Since these cross-ratios were the first numerical invariants of projective transformations to be discovered, many researchers on projective geometry thought it a praiseworthy goal to reduce all other invariants to cross-ratios, even though the reduction was sometimes very artificial. Later on we shall consider these questions more thoroughly.

These few indications will suffice to show how we can distinguish sharply between the various concepts of geometry according to their behaviour under projective transformation. Everything that remains unchanged by such transformation constitutes the subject matter of *projective geometry*, which arose during the last century, of which I have already spoken, and which we shall discuss more thoroughly later on. This name, which is used now quite generally, is better than

geometry of position (Geometrie der Lage), which was much used earlier, and by which mathematicians wished to indicate the contrast to *geometry of measure* or *elementary geometry*, which embraced all geometric properties, including those that are not invariant under projective transformation. The older name conceals entirely the fact that many metric properties, in particular the values of the cross-ratio, belong hereto. [101]

Central Projection of Space upon a Plane (Projectivity with Vanishing Determinant)

I should like to discuss now the *applications of projective transformations*, just as I did earlier with affine transformations.

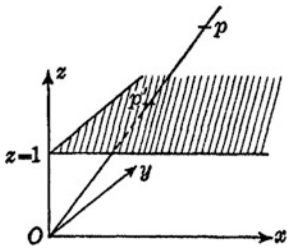

Figure 66

1. Starting with *descriptive geometry*, and making no attempt to be systematic, I shall discuss some characteristic examples.

(a) The first is the *mapping of space upon a plane by means of central perspective*, which is the direct generalization of axonometry (parallel perspective). The projecting rays proceed here from an arbitrary finite point instead of from an infinitely distant one. We select the centre of projection at the origin of coordinates O and the plane of projection as $z = 1$. (See Fig. 66.) Then, for the image $p'(x', y', z')$ of any point $p(x, y, z)$ we always have $z' = 1$, and, since p and p' lie on the same ray through O, we have

$$x' : y' : z' = x : y : z \, .$$

Hence the equations for our mapping are

$$x' = \frac{x}{z} \, , \quad y' = \frac{y}{z} \, , \quad z' = \frac{z}{z} \, .$$

This is a *special projective transformation*, and the analogy with what happens in axonometry leads us to suspect that its *determinant vanishes*. In fact, going over to homogeneous coordinates, we get

$$\rho'\xi' = \xi \, , \quad \rho'\eta' = \eta \, , \quad \rho'\zeta' = \zeta \, , \quad \rho'\tau' = \tau \, ,$$

and the determinant of the substitution is

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 0 .$$

[102] You can readily derive the various properties of this transformation by analogy with our earlier discussions, provided you note that, in general, every plane is connected with the plane of projection by a projective (two-dimensional) transformation with a non-vanishing determinant. It follows from this, in particular, that the cross-ratio of any four points on a line, or of any four rays through a point, is unchanged by the transformation.

Relief Perspective

(b) The second example concerns a projectivity, which includes the central perspective as limiting case, one with a non-vanishing determinant, the so-called *relief perspective*. The relief of an object is to be so formed that it will send the same rays to an observer's eye, placed at a definite point, which the original would send to an observer correspondingly placed. This means that, with an appropriately oriented system of coordinates, the original point and its image should lie on the same ray through the origin:

$$x' : y' : z' = x : y : z . \tag{1}$$

The difference between this and the previous case is that the original is not mapped upon a plane but is compressed into a certain narrow space segment of finite width. I assert that this is accomplished by the formulas

$$x' = \frac{(1+k)x}{z+k}, \quad y' = \frac{(1+k)y}{z+k}, \quad z' = \frac{(1+k)z}{z+k}, \tag{2}$$

which, in the first place, give at least a projectivity and also obviously satisfy equations (1). Let us form their *determinant*, using the corresponding homogeneous equations

$$\rho'\xi' = (1+k)\xi , \quad \rho'\eta' = (1+k)\eta , \quad \rho'\zeta' = (1+k)\zeta , \quad \rho'\tau' = \zeta + k\tau .$$

It will be

$$\Delta = \begin{vmatrix} 1+k & 0 & 0 & 0 \\ 0 & 1+k & 0 & 0 \\ 0 & 0 & 1+k & 0 \\ 0 & 0 & 1 & k \end{vmatrix} = k\,(1+k)^3$$

and is thus different from zero, except when $k = 0$ or $k = -1$.

For $k = 0$, (2) goes over precisely into the previous formulas of central perspective, i.e., the relief degenerates completely into a plane. The value $k = -1$ gives $x' = y' = z' = 0$, i.e., every point in space is represented by the origin, which is obviously a useless and trivial degeneration.

For the sake of definiteness, we choose $k > 0$. In order to make the transformation (2) clear geometrically, we notice, first, that every plane z = const, goes over into a parallel plane: **[103]**

$$z' = \frac{(1+k)z}{z+k}\,. \tag{3}$$

The resulting mapping of the two planes upon each other by the rays proceeding from O is perfectly intuitive, and we now need only interpret the law (3). For $z = \infty$ ($\tau = 0$), $z' = 1 + k$. *The plane parallel to the x-y-plane and at a distance $1 + k$ is the vanishing plane of the space of the image*, *and at the same time it forms*, *in a sense*, *the background of the relief upon which the infinitely distant background of the space of the object appears to be mapped*. The plane $z = 1$ plays also an important role, since object and image coincide for that plane. This follows from the fact that if $z = 1$, then $z' = 1$ also. If, now, z increases from 1 to ∞, z' increases monotonically from 1 to $1 + k$, i.e., *if we restrict ourselves to objects behind the plane $z = 1$*, *we obtain actually*, *as image*, *a relief of finite depth k*. In practice, there can and must always be such a restriction. (See Fig. 67.)

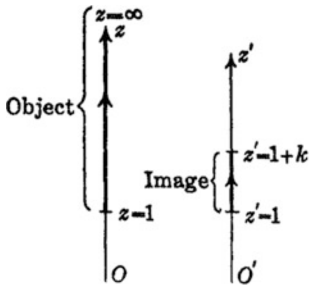

Figure 67

Examining again the relation (3), we find for the cross-ratio of the points $z, 1, z', 0$, the relation

$$\frac{z-1}{z-0}\cdot\frac{z'-0}{z'-1} = \frac{z-1}{z}\cdot\frac{(1+k)z}{k(z-1)} = \frac{1+k}{k}\,.$$

In general, two such values z and z' are correlated and form with the points 1 and 0 a cross-ratio of constant value.

We have a model in our mathematical collection, which represents, in relief perspective, a sphere on a cube, a cone of revolution, and a cylinder of revolution. Examined at the proper distance, the model actually gives a very clear impression

of the original bodies. Of course, psychological effects play an important part. The isolated fact that the same light rays enter an eye does not suffice to determine the spatial impression; habit must certainly play an important part. Indeed, since we have seen a sphere on a cube much oftener than we have seen a narrow ellipsoid on a narrow hexahedron (that is the form of the image in relief perspective), we are disposed, from the start, to refer the light impression to the first source. A closer examination of the effects that enter here may be left to the psychologists.

This will suffice to give you a first glimpse of the application of projective trans-
[104] formations to descriptive geometry. Of course, these theorems demand further consideration, and I cannot leave this field without urging you to make a thorough study of descriptive geometry, which is, I think, indispensable for every teacher of mathematics.

Application of Projecting to Derive Properties of Conic Sections

2. The second application of projective transformations of which I wish to talk is the *derivation of geometric theorems and points of view*. You will recall that we discussed affine transformations with a similar purpose (pp. [81] sqq.).

(a) We start from the theorem that *when we subject a circle to a projective transformation or to a central perspective-transformation*, *it goes over into some* "*conic section*," i.e., into the intersection by a plane of the cone whose surface is formed by the projecting rays drawn to the points of the circle. I have here a model, which shows how an ellipse, a hyperbola, or a parabola can arise in this way. (See Fig. 68.)

(b) *It follows that*, *for projective geometry*, *there is only one single conic section*, since any two can be transformed into a circle and therefore into each other. The division into ellipse, parabola, hyperbola indicates, from this standpoint, no really fundamental difference, but reflects merely the accidental position with reference to that line which is ordinarily called "infinitely distant."

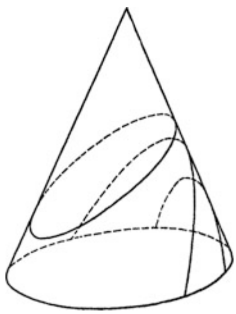

Figure 68

(c) Let us now derive the following *fundamental cross-ratio theorem for conics*: *Any four fixed points* 1, 2, 3, 4 *on a conic section are projected from a fifth movable*

point P of the same conic section by four rays whose cross-ratio is independent of the position of P.

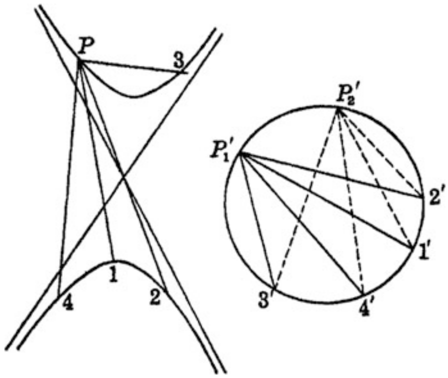

Figure 69

To prove this, we go back to the circle from which the conic section in question arose by central perspective. Since, in this, the cross-ratio is unchanged, our **[105]** theorem will be true, in general, if only we can show that the four corresponding points $1', 2', 3', 4'$ on the circle are projected from two other arbitrary points P_1', P_2', by rays, which have the same cross-ratio. But this is at once evident, for, by the theorem on inscribed angles, the angles of the family of rays $P_1'(1', 2', 3', 4')$ on the one hand, and of $P_2'(1', 2', 3', 4')$ on the other, are equal in pairs; hence the two cross-ratios formed from the sines of the angles of the two ray quadruples are equal.

(d) Steiner actually based his definition of conic sections on this theorem by starting from two "projectively related" families of rays, in which two corresponding ray quadruples have the same cross-ratio. *A conic section is then the locus of the intersections of corresponding rays of these projectively related families.*

These few remarks will suffice to make clear to you the great significance of projective transformations for the theory of conic sections. You can find a more complete account in any textbook on projective geometry.

Proceeding further in the wide range of this chapter, we shall now come to new classes of geometric transformations not belonging to the class of linear transformations which we have thus far considered and which have led us progressively from displacements to the most general projectivities.

III. Higher Point Transformations

We shall now investigate transformations that are represented, not by linear functions, but by higher *rational algebraic functions*, *or even by transcendental functions*:

$$x' = \phi(x, y, z)\,, \quad y' = \chi(x, y, z)\,, \quad z' = \psi(x, y, z)\,.$$

Adhering to the plan of this lecture course, I shall not give a systematic presentation, but I shall present a series of particular examples, which have general significance in pure mathematics and, above all, in its applications.

First of all, I shall discuss that one of these transformations, which is most frequently used: the *transformation by means of reciprocal radii*.

1. The Transformation by Reciprocal Radii

This transformation *carries each point p into that point p' on the line Op joining p with the origin O, for which the product $Op \cdot Op'$ is equal to a given constant*. (See Fig. 70.)

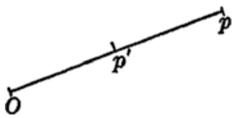

Figure 70

As you know, this transformation plays an important role in pure mathematics, and particularly in *the theory of functions of a complex variable*. It appears not less **[106]**
frequently, however, in physics and in other applications. Later on, we shall discuss at length *one* particular application.

1. In treating this transformation, I shall again start with the derivation of its equations in rectangular coordinates. Since p and p^1 lie on the same line through *0*, we have

$$(1) \qquad x' : y' : z' = x : y : z\,,$$

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_9

and from the relation between the distances Op and Op', setting the constant equal to 1, for simplicity, we find

$$(2) \qquad \left(x^2+y^2+z^2\right)\left(x'^2+y'^2+z'^2\right)=1\,.$$

Therefore, the equations of the transformation are

$$(3) \qquad x'=\frac{x}{x^2+y^2+z^2}\,,\quad y'=\frac{y}{x^2+y^2+z^2}\,,\quad z'=\frac{z}{x^2+y^2+z^2}\,.$$

In the same way, we obtain, conversely,

$$(4) \qquad x=\frac{x'}{x'^2+y'^2+z'^2}\,,\quad y=\frac{y'}{x'^2+y'^2+z'^2}\,,\quad z=\frac{z'}{x'^2+y'^2+z'^2}\,.$$

Thus not only are the coordinates of p' expressed rationally in terms of those of p, but also the coordinates of p are expressed in terms of those of p', as rational functions; and the functions that occur are the same in both cases. The denominator in each case is a quadratic expression. We have here *a particular case of what is called a quadratic birational transformation*. There is, moreover, an extensive class of such birational transformations (in general uniquely reversible), which are represented, in both directions, by rational functions. Under the name *Cremona transformations* they are the object of a widely developed theory, to which I must at least allude as I discuss the simplest one of them.

2. Equations (3) and (4) show that to every point p in space there corresponds a point p', and, conversely, to every point p' there corresponds a point p, if we except (for the present) the origin. However, if we let x, y, and z approach zero simultaneously, the denominator of (3) will vanish, of higher order than the numerator, and x', y', and z' become infinite. We could call the origin, therefore, a *vanishing point* of the transformation. Conversely, if x', y', and z' become infinite in some way, then, by (4), x, y, and z all approach zero. If, then, we were to use our earlier terminology, we should say that one single point corresponds to the whole infinitely distant plane. However, this "infinitely distant plane" was merely a convenient expression, which was suitable for the projective transformation. It in-
[107] dicated that, under that transformation, the infinitely remote part of space behaved as though it were a plane, i.e., it went over into the points of some finite plane, and this made it possible to enunciate theorems without making exceptions, and without introducing distinctions of cases. There is nothing to hinder us from employing here a different form of expression, and from stating, by means of it, for our present purpose, theorems likewise valid without exception. By our transformation, the infinitely remote in space is transformed into a point; hence we say, simply, *there is only one infinitely distant point*, *and it corresponds*, *under our transformation*, *to the origin of coordinates*. Then our transformation in fact is *uniquely reversible without exception*.

It is impossible to overemphasise that here, as well as in our earlier remarks, we are not thinking, in the remotest sense, of metaphysical representations of the true nature of infinity. There are, of course, always people, who, partial by habit

to the one or to the other form of expression, would like to assign a transcendental meaning to infinity. Such advocates of these two points of view sometimes fall into controversy. Of course they are both wrong. They forget that we are really concerned merely with an arbitrary convention, which is appropriate for the one purpose or for the other.

3. The principal property of our transformation is that (speaking generally) it *transforms spheres into spheres*. Indeed, the equation of a sphere has the form

$$A\left(x'^2 + y'^2 + z'^2\right) + Bx' + Cy' + Dz' + E = 0\,. \tag{5}$$

Substituting for x', y', z' their values in (3), replacing the quadratic term x'^2, y'^2, z'^2 by means of (2), and multiplying through by x'^2, y'^2, z'^2, we get $A + Bx + Cy + Dz + E(x^2 + y^2 + z^2) = 0$, which is indeed the equation of a sphere. To be sure, it should be noticed that the equation (5) (for $A = 0$), includes also planes, which we can appropriately consider here as *special spheres*; they are in fact those spheres, which contain the point at infinity. Under our transformation they go over into spheres that pass through the point, which corresponds to the point at infinity, that is, the origin. Conversely, any spheres that go through the origin go over into spheres through the point at infinity, that is, into planes. With this convention, the theorem that spheres correspond to spheres is valid *without exception*.

Since two spheres (likewise a sphere and a plane) intersect in a circle, it follows also furthermore that *to a circle there corresponds always a circle*, whereby, in particular, *straight lines* are included as "*circles through the point at infinity*." Conversely, to a straight line corresponds, under our transformation, a circle through the origin.

4. This last theorem is, of course, still valid if we restrict the transformation by **[108]**
reciprocal radii to a *plane*. This gives rise to an elegant solution of the problem of generating a straight line, which is very elementary and which belongs properly also to the field of interests of the non-mathematician. The problem is to guide a point, by means of a linkage of rigid rods, so that it will describe a straight line. Formerly, in the construction of steam engines, particular importance was placed upon mechanisms that would effect the transmission between the piston, which moves rectilinearly, and the end of the crank, which describes a circle.

Peaucellier's Construction

This directs our interest to the *inversor* which Charles-Nicolas Peaucellier, a French officer, constructed in 1864, and which caused a sensation then, although the construction is very simple and fairly obvious. The apparatus consists of six jointed rods. (See Fig. 71.) Two of the rods, of length l, are attached at a fixed point O; the other four rods, all of length m, form a rhombus whose opposite vertices are the ends of the rods l. Call the two free vertices of this rhombus p and p'. The apparatus has two degrees of freedom: First, one can incline the two rods l to each other at will, and, second, one can rotate them together about O. With every such motion,

however, Opp' remains a straight line, as it is easy to prove geometrically, and the product

$$Op \cdot Op' = l^2 - m^2 = \text{const.}$$

is independent of the position of p. Thus *the apparatus actually effects a transformation by reciprocal radii with O as centre*. We need only move p on a circle through O, in order to force p' – according to the theorems of no. 3 – to move actually on a straight line. This result is secured at once if we attach at p a seventh rod pC, whose other end, C, is fixed at the midpoint between O and the initial position of p. Then there remains but one degree of freedom, and *p' will, in fact, be carried along a straight line*. It should be noticed that the point p' cannot traverse the entire unlimited line, but that its freedom to move is limited by the fact that its distance from O remains always less than lm, because the given lengths of the rods do not permit more extended motion. In some models, the point C is displaced a little,
[109] so that the circle, which p traverses passes close to O, and p' moves, therefore, no more in a straight line but on a *circle of very large radius*. This application of the apparatus also may be useful at times.[44]

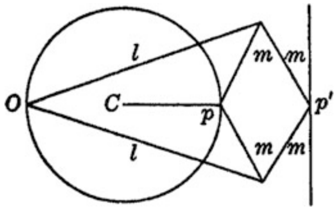

Figure 71

5. Of the general properties of the transformation by reciprocal radii, I will emphasise, lastly, that of the *preservation of angles*. This means that *the angle, which two surfaces make with each other at any point of their curve of intersection is the same before and after the transformation*. I shall omit the proof since I am not concerned, in this survey, with carrying out the details.

Stereographic Projection of the Sphere

6. We can look upon *stereographic projection*, which also plays an important role in the applications, as a special chapter of the transformations by reciprocal radii. It is obtained as follows. Let us consider the sphere which is carried by our transformation into *the fixed plane $z' = 1$*. By the third of the formulas (3) the equation of

[44] [See also A. B. Kempe, *How to Draw a Straight Line*, London, 1877; and Gerhard Hessenberg, *Gelenkmechanismen zur Kreisverwandtschaft*, Heft 6 der Naturwissenachaftlich-medezinischen Abhandlungen der Württembergischen Gesellschaft zur Förderung der Wissenschaften, Abteilung Tübingen, 1924.]

this sphere is

$$1 = \frac{z}{x^2 + y^2 + z^2},$$

which may be written in the form

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

Thus the sphere which is transformed into the plane $z' = 1$ has a *radius* $\frac{1}{2}$, *and has its centre at the point* $z = \frac{1}{2}$ on the z-axis. It passes through the origin, and is tangent to the image plane $z' = 1$. (See Fig. 72.) We can at once make clear the details of the relation between the plane and the sphere if we use the space family of rays through the centre O, and discover the corresponding points. I shall state the following theorems without proof.

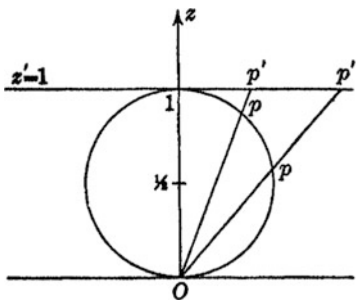

Figure 72

1. The mapping is, *without exception*, *reversibly unique*, if we think of the infinitely distant part of the plane as a point, which is then mapped upon the point O on the sphere.

2. Circles on the sphere correspond to circles in the plane; in particular, circles through O correspond to circles through the point at infinity, i.e., to straight lines.

3. The relation between the two surfaces *preserves angles*, or, as it is customary to say, *the transformation is conformal*.

You know, of course, that stereographic projection has great significance in *the theory of functions of a complex variable*. Indeed, I used it to advantage frequently in my lecture course of last semester.[45] Of other applications in which it plays an equally important role, I would mention *geography* and *astronomy*. Stereographic projection was known to the ancient astronomers; even today, you find in every atlas representations of the hemispheres, and of the polar regions of the earth, in stereographic projection. **[110]**

I shall now present a few more examples from the last-mentioned field of application.

[45] See Volume I, p. [113] sqq.

2. Some More General Map Projections

A digression in this direction seems to me especially appropriate for the present lecture course. The *theory of geographic maps* is, after all, a subject, which is of great importance in school teaching. It will interest every boy to hear from what point of view the maps in his atlas were drawn. The teacher of mathematics can put more feeling into his teaching, if he can give the desired information, than he can if he discusses only abstract questions. Thus every prospective teacher should be informed in this field, which, moreover, furnishes the mathematician with interesting examples of point transformations.

It will serve our purpose best if, at the outset, we think of the earth as projected stereographically from, say, the south pole, upon the x-y-plane. Then, with respect to that pole, any other mapping upon a ξ-η-plane will be given by the two equations $\xi = \phi(x, y)$, $\eta = \chi(x, y)$.

Among the first representations, much used in practice, are those in which *angles are preserved*. We obtain these, as is taught in the theory of functions of a complex variable, if we *think of the complex variable $\xi + i\eta$ as an analytic function of the complex variable $x + iy$*:

$$\xi + i\eta = f(x + iy) = \phi(x, y) + i\chi(x, y)\,.$$

I should like to emphasise, however, that precisely in geographic practice use is often made of *representations in which angles are not preserved*, so that conformal transformations should not be regarded, as is often done, as the only important ones.

The Mercator Projection

Under the *conformal* representations there appears prominently the so-called *Mercator projection*, which was discovered about 1550 by the mathematician Gerhard
[111] Mercator, whose real name, by the way, was the good German name Kremer. You will find mercator maps of the earth in every atlas.

The Mercator projection is determined by choosing our analytic function f as the logarithm. It is given by the equation $\xi + i\eta = \log(x + iy)$.

As mathematicians, we can at once deduce the properties of the projection from this short formula, whereas for the geographer without mathematical training, the treatment of the mercator projection is, of course, rather difficult. Introducing polar coordinates into the x-y-plane (see Fig. 73), i.e., putting $x + iy = r \cdot e^{i\varphi}$, we get

$$\xi + i\eta = \log(r \cdot e^{i\varphi}) = \log r + i\phi\,,$$

so that $\xi = \log r$, $\eta = \phi$.

We assume that the south pole of the earth is the centre of our stereographic projection. Then the origin O of the x-y-plane corresponds to the north pole of the

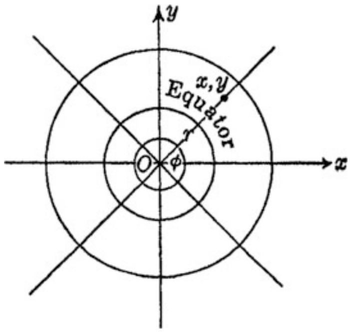

Figure 73

earth, and the rays $\phi = \text{const}$, in the x-y-plane correspond to meridians. Consequently, in the mercator projection (see Fig. 74), the meridians become $\eta = \text{const.}$, i.e., parallels to the ξ-axis. The north pole ($r = 0$, $\xi = -\infty$) lies on them to the left, the south pole ($r = +\infty$, $\xi = +\infty$), to the right, at infinity. Since the angle ϕ is undetermined to within multiples of 2π, the mapping is infinitely many-valued, and each parallel strip of width 2π, parallel to the ξ-axis, gives an image of the entire surface of the earth. The circles of latitude, $r = \text{const.}$, become, in the mercator map, the parallels $\xi = \text{const.}$, i.e., since angles are of course preserved, they are the orthogonal trajectories of the images of the meridians. To the equator ($r = 1$), there corresponds the η-axis ($\xi = 0$).

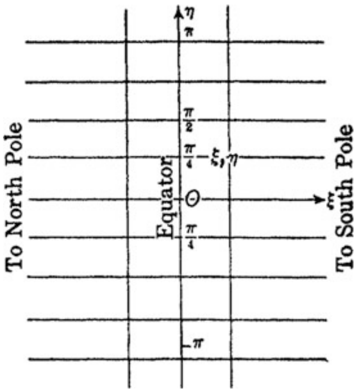

Figure 74

The Tissot Theorems

[112]

This one example may serve to arouse you to further study of the numerous transformations in the geographical theory of mapping. Let me now pass on, rather, to

a *more general theorem* of this theory. Those of you who have busied yourselves with geography have heard, certainly, of the *Tissot theorems* which Nicolas Tissot developed in his book, translated by Ernst Hammer in Stuttgart.[46] It is very easy to make its contents clear, from our standpoint.

Let there be *two geographic maps*, representations of the earth's surface upon a x-y-plane and a ξ-η-plane, each of which may be arbitrary and not necessarily conformal. The two will stand in some relation to each other, which we may write in the form $\xi = \phi\,(x, y)$, $\eta = \chi(x, y)$.

We shall examine the neighbourhood of two corresponding positions (x_0, y_0) and (ξ_0, η_0), where $\xi_0 = \phi\,(x_0, y_0)$, $\eta_0 = \chi(x_0, y_0)$. For this purpose we introduce new variables (x', y') and (ξ', η') by means of the equations

$$\begin{aligned} x &= x_0 + x' \quad & y &= y_0 + y' \\ \xi &= \xi_0 + \xi' \quad & \eta &= \eta_0 + \eta' \,. \end{aligned}$$

We obtain then, by development according to Taylor's theorem,

$$\begin{aligned} \xi' &= \left(\frac{\partial\phi}{\partial x}\right)_0 \cdot x' + \left(\frac{\partial\phi}{\partial y}\right)_0 \cdot y' + \dots, \\ \eta' &= \left(\frac{\partial\chi}{\partial x}\right)_0 \cdot x' + \left(\frac{\partial\chi}{\partial y}\right)_0 \cdot y' + \dots, \end{aligned}$$

where the derivatives are to be taken for $x - x_0$, $y = y_0$, and where terms of higher order are indicated by dots. We restrict ourselves, now, to *such a small neighbourhood of* (x_0, y_0) that the indicated linear terms give a sufficient approximation to the actual values of (ξ', η'). This means, of course, that we exclude singular positions (x_0, y_0) for which such a neighbourhood does not exist. Thus we exclude a point at which all four partial derivatives vanish simultaneously, so that the linear terms would not give a usable approximation. Then if we look at the linear equations thus obtained between (x', y') and (ξ', η'), we have at once the fundamental theorem, which forms the basis of Tissot's reflections: *Two geographic maps of the same terrain are connected*, *in the neighbourhood of a non-singular position*,
[113] *approximately*, *by an affine transformation*. If we now apply our earlier theorems on affine transformations, we obtain actually all of the so-called Tissot theorems. I shall merely remind you of a few principal points. We know that everything depends on the determinant of the affine transformations, i.e., here, on the determinant

$$\Delta = \begin{vmatrix} \left(\frac{\partial\phi}{\partial x}\right)_0 & \left(\frac{\partial\phi}{\partial y}\right)_0 \\ \left(\frac{\partial\chi}{\partial x}\right)_0 & \left(\frac{\partial\chi}{\partial y}\right)_0 \end{vmatrix},$$

which is called the *functional determinant of the functions ϕ and χ for the position* $x = x_0$, $y = y_0$. We always avoid the case $\Delta = 0$ in these applications, for in

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[46] *Die Netzenentwürfe geographischer Karten nebst Aufgaben über Abbildungen beliebiger Flächen auf einander*, Stuttgart, 1887.

that case the neighbourhood of (x_0, y_0) in the x-y-plane would be mapped upon a curve segment of the ξ-η-plane, and the geographer would hardly consider such a map as usable. We are thus to *consider here* $\Delta \neq 0$. In our earlier discussions (see pp. [78]–[79]) we made clear how such an affine transformation comes about; hence we can now take over the theorem: *The neighbourhood of the point* (ξ_0, η_0) *is obtained from that of the point* (x_0, y_0), *with the accuracy which here concerns us*, *by subjecting the latter to a pure deformation in two mutually perpendicular directions and by then rotating it about a suitable angle*. You will find in Tissot's book that he actually gives a clear ad-hoc deduction of this theorem, and you have here an interesting example of how those concerned with the applications manage to meet the mathematical needs of their own subject. To the mathematician, the thing always seems very simple, but it is still instructive for him to know what these applications require.

I shall now pass to the consideration of a last general class of point transformations.

3. The Most General Biunique Continuous Point Transformations

All of the mapping functions, which we have thus far considered were continuous and differentiable without restrictions, indeed they were analytic (to be expanded into a Taylor series). However, we admitted multiple, even infinitely many-valued functions (e.g., the logarithm). We shall now set down as precisely, our chief requirement *that our mapping functions shall be without exception reversibly unique*. *We shall assume also that they are continuous*. We shall make no assumptions, however, as to the existence of derivatives, etc. We inquire as to the properties of geometric figures, which remain unchanged under these most general reversibly biunique and continuous transformations. Let us think, say, of a surface or a solid **[114]**
made of rubber, with figures marked upon it. What is preserved in these figures if the rubber is arbitrarily distorted without being torn?

Analysis Situs

The totality of properties, which we find in the treatment of this question makes up the field that is called *analysis situs*. We might call it the *science of those properties, which depend exclusively upon position and not at all upon size*. The name comes from Riemann, who, in his famous paper of 1857, *Theorie der Abelschen Funkionen*,[47] was drawn into such investigations by function-theoretical interests.

[47] *Journal für die reine und angewandte Mathematik*, vol. 54 = *Gesammelte mathematische Werke* (2nd edition, Leipzig, 1892), p. 88. – Riemann, following Leibniz, uses here the word "analysis"

Since that time, moreover, it has often happened that analysis situs is not mentioned in books on geometry, and is left for discussion in the theory of functions when it is needed. It was not so, however, with *Möbius*, who, in a paper written in 1863,[48] discussed analysis situs from its purely geometric interest. He calls those figures, which transform into each other through biunique continuous distortion *elementarily related* figures, because the properties, which are invariant under these transformations are the simplest possible properties.

We shall restrict ourselves here to the *investigation of surfaces*. To begin with, we should note a property which was first discovered by Möbius, and which Riemann had missed entirely: the *distinction*, *namely*, *as to whether a surface is one-sided or two-sided*. Indeed we have discussed (pp. [19]–[20]) the *one-sided Möbius band*, upon which, by continuous movement, one can come unawares from the one side to the other, so that a distinction between the two sides no longer has any meaning. It is clear that this property persists through all continuous distortions and that therefore, *in analysis situs*, *we must actually distinguish*, *from the beginning*, *between one-sided and two-sided surfaces*.

For the sake of simplicity we shall concern ourselves here *only with two-sided surfaces*, especially since they alone are ordinarily considered in the theory of functions of a complex variable. However, the theory of one-sided surfaces is not essentially more difficult. It turns out that for a surface, in the sense of analysis situs, there are two *natural numbers* which are *completely characteristic*: The *number μ of its boundary curves* and the *number p of closed cuts which do not*
[115] *separate it into parts* (the so-called genus). More precisely, *a necessary and sufficient condition that two two-sided surfaces be applicable to each other biuniquely*, *and continuously* (*that they be "elementarily related" or*, *as we say today*, *they be homeomorphic*) *is that these two numbers μ and p shall be the same for both surfaces*. The proof of this theorem would carry us too far afield. I can merely illustrate these numbers μ and p by a *few examples*.

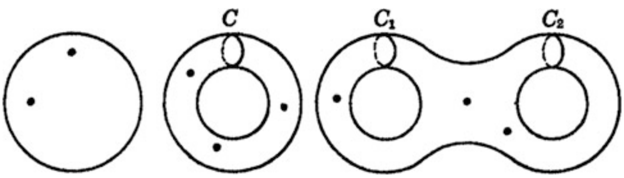

Figure 75

Let us think of *three surfaces* placed alongside of one another, a *sphere*, a *torus*, and a *double torus* (shaped like a pretzel), as they appear schematically in Fig. 75. Each is a closed surface, i.e., it has no boundary curve; hence $\mu = 0$. In the first

in its original methodological sense, not with the meaning, which it has taken on as a mathematical term.

[48] *Theorie der elementaren Verwandtschaft*, Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, mathematischphysikalische Klasse, vol. 15, p. 18 ff. = *Gesammelte Werke*, vol. 2 (Leipzig, 1886), p. 433 ff.

example, every closed cut divides the surface into two separate parts, so that $p = 0$. In the second example, a meridian curve C represents a closed cut which does not separate the surface into parts. After the curve C has been drawn, however, every additional closed cut actually divides the surface into parts. This is precisely what we mean when we say $p = 1$. In the third example, $p = 2$, as is shown by the two different meridian curves C_1 and C_2, on the two separate handles. By the addition of more handles, we can create surfaces with any desired value of p. On the other hand, we can give μ any desired integer value different from zero by making in these surfaces small holes or *punctures*, each of which provides a boundary curve. Thus we can actually set up surfaces with arbitrary values of p and μ, and all other surfaces with the same values of p and μ must then be homeomorphic with them, no matter how different they may be in appearance. The theory of functions offers many examples of such surfaces.

I must explain here also the term *connectivity*, which Riemann introduced. By it he means the number $2p + \mu$, and he calls the surface $(2p + \mu)$*-ply connected*. A surface is *simply-connected if* $2p + \mu = 1$, so that $p = 0$ and $\mu = 1$; that is, it is homeomorphic to a sphere with one puncture, which we could deform continuously into a *circular disk* by enlarging the hole. (See Fig. 76.)

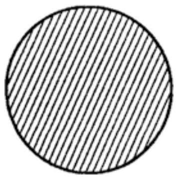

Figure 76

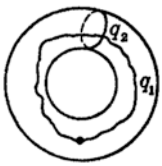

Figure 77

Riemann also introduces the notion of *crosscut*, i.e., a cut, which joins one boundary point with another. Thus we can speak of crosscuts only if boundary **[116]**
curves actually exist, that is, only if $\mu > 0$. We can then prove the following theorem. *Each crosscut reduces the connectivity by* 1, *so that*, *in particular*, *any surface for which* $\mu > 0$ *can be changed into a simply-connected surface by* $2p + \mu - 1$ *crosscuts*. Let us consider a torus (see Fig. 77) with one puncture ($p = \mu = 1$), and let us draw the first crosscut q_1 from this puncture and necessarily back to the same puncture. Then let us draw the second crosscut q_2, which starts and also ends in the first cut and resembles precisely the closed cut in the torus of Fig. 76. Then the connectivity is actually reduced from $2 \cdot 1 + 1 = 3$ to 1.

As to literature concerning analysis situs, there is a comprehensive list, not merely for surfaces, but also for arbitrarily extended configurations, in the *Enzyklopädie der mathematischen Wissenschaften* in the report by Max Dehn and Poul Heegard (III AB 3), which is, to be sure, very abstract. It would be highly desirable to have a more readable presentation, which would be accessible to the beginner, and in which the abstract theory would be preceded by a development of the general ideas with simple examples.[49]

Euler's Polyhedron Theorem

Analysis situs finds applications in physics, especially in potential theory. But it reaches also into *school instruction*, in *the polyhedron theorem of Euler*, concerning which I shall say a word. Euler observed that if *an ordinary polyhedron* has E vertices, K edges, and F faces *we always have the relation*

$$E + F = K + 2\,.$$

Now if we deform the polyhedron in any way which is biunique and continuous, these numbers, and hence the equation, will remain unchanged, so that the latter will still hold when E, F, K are the numbers of vertices, faces, and edges of an *arbitrary division of the sphere* or, indeed, of any *surface homeomorphic to it*, provided only that *each subdivision is simply-connected*. We can generalise this theorem at once to surfaces of arbitrary genus, as follows. *If we divide a surface which admits p closed cuts without dismemberment, into F simply-connected parts of surfaces by means of K line segments, so that E vertices are created, then we shall have*

$$E + F = K + 2 - 2p\,.$$

[117] I leave it to you to set up illustrative examples and to ponder over the proof of the theorem, or to read it in the Dehn-Heegard report. Of course, there are still broader generalisations of this theorem.

With this we shall leave eventually the theory of point transformations, and we shall try to obtain a view of the most important classes of those transformations, which transform points into space elements of another kind.

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[49] [A more recent work is Béla von Kerékjartó, *Vorlesungen über Topologie* (vol. 1, only, has appeared), Berlin, Springer 1923. Another article on analysis situs will appear soon in the *Enzyklopädie der mathematischen Wissenschaften*, by Heinrich Tietze.]

IV. Transformations with Change of Space Element

1. Dualistic Transformations

The most obvious cases are those correspondences, which interchange *point* and *line* in a two-dimensional region, or *point* and *plane* in a three-dimensional region. I shall restrict myself to the first case, and I shall follow the line of thought, which Plücker first used in 1831 in the second part of his *Analytisch-geometrische Entwickelungen*, which we mentioned earlier (p. [61]). For it, the analytic formulation constituted the point of beginning.

The first idea used by Plücker, as I have discussed already (pp. [63]–[64]), is to place on an equal footing with ordinary coordinates the constants u and v in the equation of the straight line,

$$ux + vy = 1\,, \tag{1}$$

to regard u and v as *line coordinates*, and to build up the structure of analytic geometry by using these two sorts of coordinates in analogous "dual" ways. Thus, in the plane, there correspond to each other the *curve as a locus of points given* by the point equation $f(x, y) = 0$, and the *curve as the envelope of a singly infinite family of straight lines, defined* by the line equation $g(u, v) = 0$. A proper *transformation*, such as we now wish to consider, will be obtained, of course, only when we add to our plane E a second plane E', and set up a relation between the line coordinates u and v in E and the point coordinates x' and y' in E'. Thus the most general transformation of this kind would be given by the two equations

$$u = \phi(x', y')\,, \quad v = \chi(x', y') \tag{2}$$

i.e., to each point (x', y') in E' there will correspond the line in E whose equation is obtained by substituting these values (2) in (1).

1. To begin with, let us consider the *simplest example of such a transformation*, which is given by the equations

$$u = x'\,, \quad v = y'\,. \tag{3}$$

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_10

[118] By means of this transformation, to the point (x', y') in E', there will correspond in E the line

(3a) $$x'x + y'y = 1\,.$$

If we now superimpose the planes E and E' so that their coordinate systems coincide, we see that this equation represents the polar of the point (x', y') with respect to the unit circle about the origin, $(x^2 + y^2 = 1)$, so that our *transformation is the familiar polar relation for the circle*. (See Fig. 78.)

We notice that, in place of the two equations (3), the *one* equation (3a) suffices to define the relation, since it is the equation of the line corresponding to any point (x', y'). Since it is completely symmetrical in x and y on one hand and in x' and y' on the other, *the two planes E and E' must play the same role in our relation*, i.e., to every point in E there must also correspond a line in E'. It makes no difference, when the two planes coincide, whether we think of the point as in E or in E'.

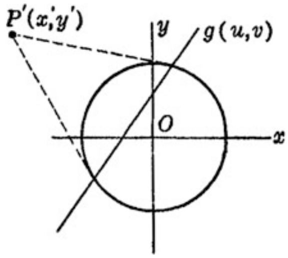

Figure 78

With respect to the first property, we call the transformation *dual in the narrower sense*; with respect to the second, *reciprocal*. Thus, without making any distinction between the two planes, we can speak simply of the correspondence of a definite polar to a pole, and then express the reciprocal property in the manner stated on p. [62].

As for other properties of this transformation, I remark merely that, to a curve traced by the point (x', y') in the plane E', there would correspond, by the principle of duality, the curve in the plane E enveloped by the corresponding line (u, v).

2. By analogy with our earlier discussion of the most general "collineation," it can be proved easily that the *most general dual relation* is obtained if we generalise the assumption (3) and set u and v equal to *linear fractional functions of x' and y' with the same denominator*:

(4) $$\begin{cases} u = \dfrac{a_1x' + b_1y' + c_1}{a_3x' + b_3y' + c_3}\,, \\[2ex] v = \dfrac{a_2x' + b_2y' + c_2}{a_3x' + b_3y' + c_3}\,. \end{cases}$$

Substituting these values for u and v in (1), multiplying by the common denomina-
[119] tor, and noting that the nine coefficients $a_1, \ldots, c_3$ are arbitrary, we obtain the *most*

general linear equation in x and y as well as in x' and y':

(4a) $$a_1xx' + b_1xy' + c_1x + a_2yx' + b_2yy' + c_2y - a_3x' - b_3y' - c_3 = 0\,.$$

Conversely, every such "bilinear" equation in x, y and x', y' represents a dual transformation between the planes E and E'. For, if we assume that one pair of coordinates are constant, i.e., if we think of a fixed point in one of the planes, the equation is linear in the other two coordinates and represents a *straight line* in the other plane, corresponding to that fixed point.

3. This relation, however, is not in general *reciprocal* in the sense defined above, unless two symmetrical terms in (4a) always have the same coefficient, in which case the equation is

(5) $$Axx' + B(xy' + yx') + Cyy' + D(x + x') + E(y + y') + F = 0\,.$$

The transformation thus determined is familiar from the theory of conic sections. It expresses the *correspondence of pole and polar with respect to the conic section whose equation is*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0\,.$$

Every such polar relation is dual and reciprocal.

We can pass immediately from this to the consideration of an essentially more general class of transformations with a change of the space element, namely, the contact transformations.

2. Contact Transformations

These transformations, so named by Sophus Lie, are obtained if, instead of the bilinear equation (4a), we start with an arbitrary higher equation in the four point coordinates of the two planes:

(1) $$\Omega(x, y; x', y') = 0\,.$$

We shall assume that this equation satisfies the requisite conditions of continuity. It is called, after Plücker, the *aequatio directrix* or *directrix equation*. For plane geometry, all the relevant developments are found in Plücker's work mentioned above.[50] To begin with, we keep x and y fixed, i.e., we consider a definite point $P(x, y)$ in E. (See Fig. 79.) Then the equation $\Omega = 0$ represents, in the variable coordinates x' and y', a definite curve C' in the plane E', and we make this curve **[120]**
correspond, as a new element of the plane E', to the point P, as we did earlier with the straight line. If, however, we now take a fixed point $P'(x', y')$ in E', say on

[50] *Loc. cit.*, pp. 259–265.

the curve C', then the same equation $\Omega = 0$, in which we now think of x' and y' as fixed and of x and y as variable coordinates, represents a definite curve C in E. Of course, the curve C must pass through the first point P. In this way, we have established a correspondence between the points P in E and the ∞^2 curves C' in E', and between the points P' in E' and the ∞^2 curves C in E, just as we established earlier a correspondence between points and straight lines.

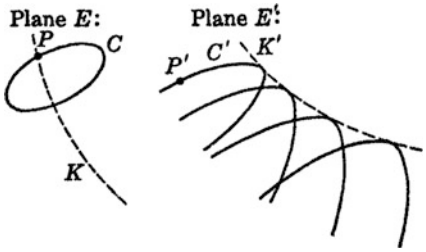

Figure 79

If, now, a point P in E moves on an arbitrary curve K (indicated by a broken line), there will correspond to each position of P a definite curve C' in E'. In order to obtain from the simply infinite family made up of the curves C', a single curve in E' which we can set into correspondence with the curve K in E, we apply to the present case the *envelope principle* already used in the relation of duality: *We place in correspondence with K that curve K' in E' which is enveloped by the curves C' that correspond to the points of K by means of the equation* $\Omega = 0$. Evidently, we could repeat the same argument, starting with an arbitrary curve K' in E'. Thus we have finally derived from the directrix equation $\Omega = 0$, a transformation of the two planes by which to every curve in the one plane, there corresponds a definite curve of the other plane.

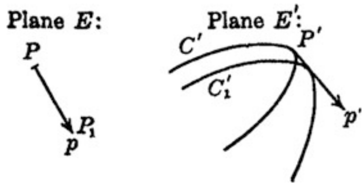

Figure 80

In order to follow this discussion *analytically*, let us replace the curve K by a rectilinear polygon with very short sides, as we habitually do in differential calculus for the sake of clearness, and let us ask what corresponds to a single such polygonal side. We always have in mind, of course, a passage to the curve as a limit, so that by the polygon side we understand, really, *a point P and its direction of motion* (the direction of the tangent to K at P); together these form a so-called *line element*. We now choose, in this direction from P, a point P_1 (see Fig. 80) with coordinates

$x+dx$ and $y+dy$, where dx and dy are small and are ultimately to approach zero, but where dy/dx always has the definite value p which characterises the given direction at P. To the point P' corresponds the curve C' in E' whose equation in the running coordinates x' and y' is

$$\Omega(x, y; x', y') = 0\,.$$

But to the point P_1 there corresponds the curve C_1' whose equation is **[121]**

$$\Omega(x + dx, y + dy; x', y') = 0\,,$$

or, expanding in terms of dx and dy, and retaining only linear terms because of the ultimate passage to the limit, we obtain

$$\Omega\left(x, y; x', y'\right) + \frac{\partial \Omega}{\partial x} dx + \frac{\partial \Omega}{\partial y} dy = 0\,.$$

These two equations give the coordinates x' and y' of the intersection of C' and C_1', which, in the limit, is the point of contact of C' with the envelope K'. Since $dy/dx = p$, we may write these equations in the form

$$\text{(2)} \qquad \left\{ \begin{array}{l} \Omega\left(x, y; x', y'\right) = 0\,, \\ \frac{\partial \Omega}{\partial x} + \frac{\partial \Omega}{\partial y} p = 0\,. \end{array} \right.$$

Moreover, C' and C_1' have, in the limit, a common tangential direction in P' given by the equation $dy/dx' = p'$, which is also the direction of the envelope K' in P'. Since $\Omega = 0$ is the equation of C' in the running coordinates x' and y', this tangent direction is determined by the equation

$$\frac{\partial \Omega}{\partial x'} dx' + \frac{\partial \Omega}{\partial y'} dy' = 0$$

or

$$\text{(3)} \qquad \frac{\partial \Omega}{\partial x'} + \frac{\partial \Omega}{\partial y'} p' = 0\,.$$

Thus, if we know a point P of K and the direction p of the tangent at P, then a point P' on the corresponding curve K' is determined, together with the direction p' at P'. We say, therefore, that *our transformation establishes a correspondence between every line element* x, y, p *of the plane E and a definite line element* x', y', p' *of the plane E*', *by means of equations* (2) *and* (3).

If we apply this argument to each side of the polygon, which approximates the corresponding curve K (or to each of the line elements of K), we get in E' the sides of the polygon which approximates the corresponding curve K' (or the line elements of K'). Hence *the equations* (2), *solved for* x' *and* y', *give the analytic*

representation of the curve K', when we let x, y, and p, the coordinates and the slope, run through the values given by all the points on K. (See Fig. 81.)

It now becomes clear why Lie called these transformations *contact transfor-*
[122] *mations*. For, if two curves in E touch each other, this means that they have a line element in common; hence the corresponding curves in E' must have a common line element, i.e., a common point and a common direction through that point. *The tangency of two curves is thus an invariant under the transformation*, which is what the name implies. Lie developed extensively the theory of these contact transformations also for *space*. He began in 1896, together with Georg Scheffers, a comprehensive presentation in his work entitled *Geometrie der Berührungstransformationen*, which unfortunately was not continued much beyond the first volume.[51]

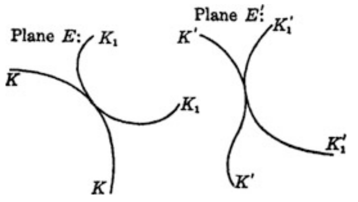

Figure 81

Having given this brief discussion of the *theory* of transformations with a change of the space element, I shall try to enliven it with a few concrete examples, in order to show what can be done with these things in the applications.

3. Some Examples

Shape of Algebraic Order and Class Curves

Let me speak first of the *dual transformations* and of the role, which they play in the *theory of the forms of algebraic curves*. We shall inquire how typical curve-forms change under dual transformation, as in the reciprocal polar relation with respect to a conic section. We must restrict ourselves, of course, to a few characteristic cases. Thus I shall examine first, under *curves of third degree*, the type which has an *odd number of branches*, and which is cut by every straight line either in one or in three real points. In the adjacent sketch (Fig. 82) there is one *asymptote*; but we can immediately obtain from this a form with three *asymptotes* by transforming the curve projectively so that a line, which cuts it in three points, is thrown to infinity. In any event, the curve has three real points of *inflection*, and these have the special property of being *collinear*. By dualisation of this curve, we get a *curve of class*

[51] Vol. 1, Leipzig, 1896. The first three chapters of the second volume appeared posthumously in *Mathematische Annalen*, vol. 59 (1904).

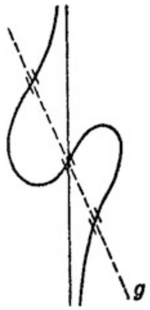

Figure 82

three, to which there can be drawn from any point either one or three tangents. To the point of inflection there must correspond a *cusp*, as will become clear upon careful reflection. Moreover, you will find these matters discussed thoroughly in my earlier lecture courses on geometry. The curve of the third class, which arises here (Fig. 83), has thus three cusps, and the *tangents at those cusps must go through a point P'* which corresponds to the line g on which the three points of inflection lie. **[123]**

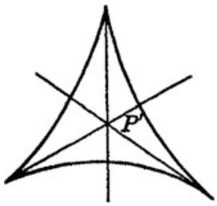

Figure 83

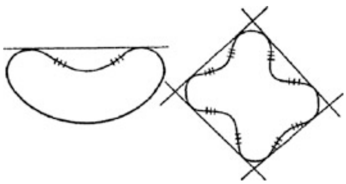

Figure 84

I shall now make similar brief statements concerning the *curves of degree four and those of class four*. A curve of fourth degree can appear in the form of an *oval with an indentation*; indeed, there exist also *forms with two*, *three*, *or four indentations*. (See Fig. 84.) In the first case, there will be two real *points of inflexion* and one *double tangent*; in the others there can be as many as *eight inflexions* and

four *double tangents*. If we dualise, we must add to what was said above that the dual of a double tangent is a *double point*. There will arise, therefore, types of *curves of fourth class* with from two to eight *cusps* and from one to four *double points*, as sketched in Fig. 85. There is a special charm in carefully working out the forms of algebraic curves. Unfortunately, I cannot here follow them in more detail and I must content myself with these brief indications.[52] These examples amply illustrate, however, how duality transformations bring under the same law things which at first glance seem as unlike as possible.

Figure 85

Application of Contact Transformations to the Theory of Cog Wheels

I come now to the *applications of the theory of contact transformations*. It turns out here, interestingly enough, that the idea of contact transformations, like most really good theoretical ideas, has a wide field of application. Indeed, mathematicians were making use of them long before the theory was worked out. It is the old *doctrine of cog wheels*, *or gears*, that I now have in mind particularly. It constitutes a *special chapter of kinematics*, of the general science of the mechanisms of motion, which is of central importance, for example, in the construction of machines. The devices for drawing a straight line, of which we recently discussed an example, also belong to kinematics. What I have so often said in this lecture course holds
[124] likewise here: I can of course only pick out small parts of each discipline and endeavour to make their meaning and significance as obvious as possible by means of simple examples. With the stimulation that I have supplied, I trust that you will try to fill in the details from special presentations. As chief means of orientation in the whole field of kinematics, I recommend the report by Arthur Schoenflies in the *Enzyklopädie* (IV_3), which also gives information concerning the extensive literature.

The *problem* of constructing gears is to *transfer uniform motion from one wheel to another*. However, since forces are also to be transferred at the same time, it is not enough to let the wheels roll upon each other (see Fig. 86). It is necessary to

[52] [See Felix Klein, *Gesammelte mathematische Abhandlungen*, vol. 2, pp. 89 sqq., pp. 136 sqq., pp. 99 sqq., Berlin, Springer, 1922, the two papers *Über eine neue Art Riemannscher Flächen* and the first paper *Über den Verlauf der Abelschen Integrale bei den Kurven 4. Grades*.]

provide one of the wheels with projections (teeth), which fit into depressions on the other. The problem is, therefore, to form the profiles or faces of these teeth so that *uniform rotation of the one wheel will bring about uniform rotation of the other*. That is certainly a very interesting problem, even from the geometric side. I shall give the most important part of its solution. *The teeth of one of the wheels can be chosen*, *in the main*, *arbitrarily*, with restrictions imposed by practical usability, such as that the individual teeth should not collide with one another. *The teeth of the second wheel are then necessarily fully determined*, *and*, *in fact*, *they are derived from the teeth of the first wheel by a definite contact transformation*.

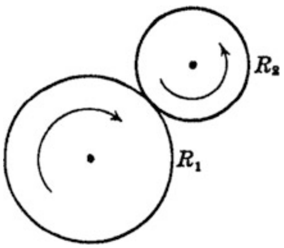

Figure 86

I need only explain briefly how this theorem comes about, without giving a full proof. We note first that we are concerned only with the motion of the two wheels relatively to each other. We may think, therefore, of one of them R_1 as fixed, while the other R_2, in addition to its own rotation, travels around R_1. Thus every point on R_2 describes in the fixed plane of R_1 an epicycloid (see Fig. 87), which is prolate, has cusps, or is intertwined, according as the tracing point is inside, on, or outside the circumference of R_2. *It follows that to every point of the moving plane of* R_2 *there corresponds a definite curve in the fixed plane of* R_1. If we derive, by the **[125]**
method already discussed, the *contact transformation* from the equation, which expresses this correspondence, we shall have precisely the contact transformation for the gears in question. It is easy to show that two curves, which correspond to each other under this transformation, actually mesh into one another in this motion.

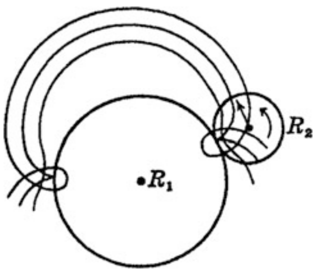

Figure 87

Finally, a word as to how the theoretical principle, thus outlined, actually takes form in the practical construction of gears. I shall mention only the simplest case, the toothing of the driving pinion. Here the teeth of R_2 are simply *points* (see Fig. 88) or, rather, since points could not transfer force, small circular pivots, the pinions. To every such small circle there corresponds, under the contact transformation, a curve, which differs only slightly from an epicycloid, namely, a curve parallel to it and distant from it by the radius of the pinion. The circles roll upon these curves when R_2 turns, so that these curves are the flanks of the teeth, which must be erected upon R_1 in order that the circular teeth of R_2 may clutch properly. In this model, which I show you, the beginnings of these curves can be seen realised as profiles of the teeth of R_1, each curve being of such width that one tooth after another clutches.

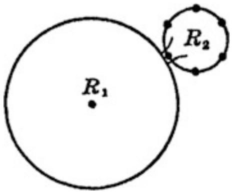

Figure 88

I show you also the realisations of two other types of gear teeth, which are much used in practice, the *involute and the cycloid gear teeth*.[53] For the first type, the tooth profiles of both wheels are involutes of circles (see Fig. 89), curves, which arise when a thread is unwound from a circle, and whose evolutes are therefore circles. For the second type mentioned, the teeth are made up of arcs of cycloids.

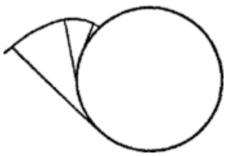

Figure 89

I hope that I have succeeded in giving you at least a preliminary orientation concerning the problems with which the theory of transformations with a change of the space element is concerned. Before we leave this second major part concerning transformations, I must supplement what I have said by a discussion of an important chapter, which should not be omitted in a cyclopedia of geometry, namely, the use of imaginary elements.

[53] All these models axe made by F. Schilling (firm of M. Schilling, Leipzig).

V. Theory of the Imaginary [126]

As you know, the theory of imaginary quantities was first developed in algebra and analysis, especially in the theory of equations and in the theory of functions of a complex variable, where, indeed, it has celebrated its greatest triumph. In addition to this, however, at an early date, mathematicians had assigned to the variables x and y in analytic geometry *complex values* $x = x_1 + ix_2$, $y = y_1 + iy_2$, and had thus added to the real points a large manifold of *complex points* without, at first, assigning any proper geometric meaning to this manner of speaking, which had been borrowed from analysis.

The Imaginary Circular Points and the Imaginary Spherical Circle

The usefulness of this new introduction was, of course, that it made superfluous those distinctions of cases, which were imposed by a restriction to real variables, and that it made it possible to enunciate theorems in a general way, without exceptions. Entirely analogous considerations in projective geometry led us to the introduction of infinitely distant points as well as the infinitely distant line and plane. What we did is appropriately called the "*adjoining of improper points*" to the proper points of space, which are conceived intuitively.

We shall now undertake both extensions at the same time. To that end, we shall introduce, as before, *homogeneous coordinates*. Remaining, for the present, in the plane, we put $x : y : 1 = \xi : \eta : \tau$ and we admit *complex values* for ξ, η, τ. We exclude the system of values $(0, 0, 0)$. Let us consider now, for example, a homogeneous *quadratic equation*

$$(1) \qquad A\xi^2 + 2B\xi\eta + C\eta^2 + 2D\xi\tau + 2E\eta\tau + F\tau^2 = 0\,,$$

and let us call the totality of systems (ξ, η, τ) which satisfy it (no matter whether they represent finite or infinitely distant points) a *curve of second degree*. The term *conic section* is sometimes used, but this can lead to misunderstanding, if not by those who know the subject, at least by those who are not familiar with the consideration of imaginary elements. The curve, under this definition, need not have one single real point.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_11

We now combine (1) with a *linear equation*

$$(2) \qquad \alpha\xi + \beta\eta + \gamma\tau = 0\,,$$

which we look upon as the definition of a *curve of first degree*, i.e., a *straight line*. These equations then have just two triplets of values $(\xi : \eta : \tau)$ in common, i.e., a *curve of the first degree and one of the second degree intersect always in two points*, *which may be real or complex*, *at a finite or at an infinite distance*, *separate or coincident*. To be sure, degenerations are thinkable, which would furnish excep-
[127] tions to this theorem. If the left side of (1) breaks up into two linear factors, one of which is identical with (2), i.e., if the curve of second degree is a pair of straight lines, and if (2) is identical with one of them, then every point of (2) is a common point. This amounts to saying that the quadratic equation, which we get by eliminating one variable from the two given equations has only vanishing coefficients. Other degenerations appear, of course, when the left side of one of the given equations, or, indeed, of both of them, vanishes identically ($A = B = \ldots = F = 0$, or $\alpha = \beta = \gamma = 0$). However, I shall ignore all such particular situations as being essentially trivial. Passing to the consideration of *two curves of second degree*, we may then enunciate the theorem that they always have *four common points*.

Let us now introduce homogeneous coordinates $x : y : z : 1 = \xi : \eta : \zeta : \tau$ also in space, and let us assign to them arbitrary complex values, excluding the system of values $(0 : 0 : 0 : 0)$. The totality of solutions of a linear homogeneous equation in these four variables is called a *surface of the first degree* (a *plane*); of a quadratic homogeneous equation, a *surface of second degree*. Then, if we ignore trivial exceptions, it is true that, in general, a *surface of second degree is cut by a plane in a curve of second degree*; and *that two surfaces of second degree intersect in a space curve of order four*, which *itself is cut by any plane in four points*. In this it is left undetermined whether or not these curves of intersection have real branches, and whether they lie wholly in a finite region or at an infinite distance.

In his *Traité des propriétés projectives des figures*, Poncelet had already applied these notions, as early as 1822, to *circles and spheres*. To be sure, he did not use homogeneous coordinates and the precise formulations, which they make possible. Instead, he followed his strong feeling for geometric continuity. In order to become acquainted with his remarkable results in exact form, let us start with the *equation of the circle*

$$(x - a)^2 + (y - b)^2 = r^2$$

which we shall write in the homogeneous form

$$(\xi - a\tau)^2 + (\eta - b\tau)^2 - r^2\tau^2 = 0\,.$$

The intersection with the line at infinity $\tau = 0$ will thus be given by the equations

$$\xi^2 + \eta^2 = 0\,, \quad \tau = 0\,.$$

The constants a, b, and r, which characterise the preceding circle, do not appear in this result. Hence, *every circle cuts the line at infinity in the same two fixed points*:

$$\xi : \eta = \pm i\,, \quad \tau = 0\,,$$

which are called the *imaginary circular points*. In the same way one can show *that every sphere cuts the plane at infinity in the same imaginary conic*: **[128]**

$$\xi^2 + \eta^2 + \zeta^2 = 0\,, \quad \tau = 0\,,$$

which is called the *imaginary spherical circle*.

The converse is also true: *Every curve of second degree, which passes through the imaginary circular points in its plane is a circle*; *and every surface of second order, which contains the imaginary spherical circle is a sphere*. These are, then, characteristic properties of the circle and the sphere.

I have purposely avoided using the expressions "infinitely distant" circular points and "infinitely distant" spherical circle, which are sometimes used. Indeed, the *distance from the origin to the imaginary circular points* is not definitely infinite, as might perhaps at first be believed. Instead, that distance has the form $\sqrt{x^2 + y^2} = \sqrt{\xi^2 + \eta^2}/\tau = 0/0$, and is therefore *indeterminate*. Any desired limiting value may be assigned to it according to the way in which we approach the imaginary circular points. Similarly, the distance from *any* finite point to the imaginary circular points is indeterminate, and the same is true of the distance from any point in space to a point of the imaginary spherical circle. This is not surprising, for we have required of these imaginary circular points that they should be at a distance r from a finite point (lie on the circle with an arbitrarily given radius r), and at the same time that they should be at an infinite distance from it. This apparent contradiction can be relieved in the analytic formula only by its yielding this indeterminateness. It is necessary to make these simple things clear, especially since untruths are often spoken and written about them.

The imaginary circular points and the imaginary spherical circle make it possible *to include the theory of circles and spheres very elegantly under the general theory of configurations of the second degree*, whereas, in the elementary treatment, certain differences seem to exist. Thus, in elementary analytic geometry, it is customary to speak always of only *two* points common to two circles, since the elimination of one unknown from their equations leads to only one quadratic equation. The elementary presentation takes no account of the fact that the two circles have in common also the two imaginary circular points on the line at infinity. The preceding general theorem actually furnishes us *four intersections*, the requisite number for two curves of the second degree. Similarly, it is customary to speak always of only *one* circle in which two spheres meet, and moreover that one may be real or imaginary. However, we know now that the spheres have in common also the imaginary *spherical circle* on the plane at infinity, and this, together with that finite circle, makes the curve of **[129]**
order four in which the general theorem requires them to intersect.

Imaginary Transformation

In this connection, I should like to say a few words about the so-called *imaginary transformation*. By this is meant a *collineation with imaginary coefficients,* which carries imaginary points in which we are interested over into real points. Thus, in the theory of the imaginary circular points, we can use to advantage the transformation

$$\xi' = \xi\,, \quad \eta' = i\eta\,, \quad \tau' = \tau\,.$$

This transformation sends the equation $\xi^2 + \eta^2 = 0$ into the equation $\xi'^2 - \eta'^2 = 0$ and *changes the imaginary circular points* $\xi\colon \eta = \pm i\,, \tau = 0$ *into the real infinitely distant points*

$$\xi' : \eta' = \pm 1\,, \tau = 0\,,$$

which are the points at infinity in the two directions that make an angle of 45° with the axes. Thus all circles are transformed into conic sections, which go through these two real infinitely distant points, i.e., into *equilateral hyperbolas – whose asymptotes make an angle of* ±45° *with the axes*. (See Fig. 90.) By means of the picture of these hyperbolas, all of the theorems on circles can be explained. This is very useful for some purposes, especially for the corresponding developments in space. I must content myself with these brief remarks if I am not to overstep the limits of this lecture course. More complete discussions are given in the lecture courses and books on projective geometry.

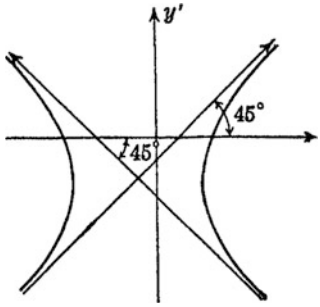

Figure 90

Staudt's Interpretation of Conjugate Imaginary Configurations

The question arises now as to whether or not a *pure geometric* method might succeed in approaching these imaginary points, planes, conic sections, etc., without drawing them by force from the formulas of analysis, as we have done thus far. The older geometers, Poncelet and Steiner, were never clear on this point. To Steiner, imaginary quantities in geometry were still "ghosts", which made their effect felt in

some way from a higher world without our being able to gain a clear notion of their existence. It was von Staudt who first gave a complete answer to the question, in his works *Geometrie der Lage*[54] and *Beiträge zur Geometrie der Lage*,[55] which we have mentioned before. We must now give some attention to his reflections. These books of von Staudt are quite hard to read, since his theories are developed at once deductively in their final form without reference to analytic formulas and without **[130]**
inductive hints. One can grasp with comfort only the genetic presentation, which follows the path probably taken by the author in the development of his ideas.

The two works of von Staudt correspond to two different steps in the development of his ideas, which I shall now present briefly. The work of 1846 is concerned primarily with the consideration of *configurations of order two with real coefficients* – I say configurations, because I wish to leave undetermined the number of dimensions (straight line, plane, or space). Let us consider, say, a *curve* of the second degree in the plane, i.e., a homogeneous quadratic equation in three variables with real coefficients:

$$A\xi^2 + 2B\xi\eta + C\eta^2 + 2D\xi\tau + 2E\eta\tau + F\tau^2 = 0\,.$$

For the analytic treatment, it is a matter of indifference whether or not this equation has real solutions, i.e., whether or not the curve of the second degree has a real branch or has only complex points. The question is by what intuitive concepts the pure geometer, in the latter case, should understand such a curve; how he should define it by geometric means. The same question arises in the one-dimensional region, when we cut the curve by a straight line, say by the x-axis $\eta = 0$. The intersections, whether they are real or not, are then given by the equation with real coefficients

$$A\xi^2 + 2D\xi\tau + F\tau^2 = 0$$

and the question is whether or not, in the case of complex roots, one can attach some geometric meaning to them.

Von Staudt's idea is, in the first place, as follows. He considers, instead of the curve of second degree, its *polar system*, which we have discussed (p. [119]), i.e., a dual reciprocal relation given by the equation

$$A\xi\xi' + B(\xi\eta' + \xi'\eta) + C\eta\eta' + D(\xi\tau' + \xi'\tau) + E(\eta\tau' + \eta'\tau) + F\tau\tau' = 0\,.$$

Because of the reality of the coefficients, this is a *thoroughly real relation*, *which creates a correspondence between every real point*, *and a real line*, whether the curve itself is real or not. The polar system, on the other hand, completely determines the curve as the *totality of those points, which lie on their own polars*. The question is left open as to whether or not such points have an existence in the real domain. In any case, however, *the polar system supplies always a real representative of the curve of second degree defined by the preceding equation*, and one, which can be used, instead of the curve itself, as the object of the investigation.

[54] Nürnberg, 1846.
[55] Nürnberg, 1856–1860.

If we now cut the curve by the x-axis, i.e., set η and η' equal to zero, we have on it, by analogy, *a one-dimensional always real polar relation*, given by the equation

$$A\xi\xi' + D(\xi\tau' + \xi'\tau) + F\tau\tau' = 0\,,$$

[131] which always sets two real points in reciprocal relation to each other. The intersections of the x-axis with the curve are the two self-corresponding points in this polar relation, the so-called *fundamental or order points*. They can be real or imaginary, but they will be only of secondary interest; the chief thing is, again, the polar relation as their real representative.

To designate the two points $(\xi/\tau, \xi'/\tau')$ which correspond to each other in such a one-dimensional polar relation, we use the expression *point pairs in involution*, which originated with Girard Desargues in the seventeenth century, and we distinguish *two principal kinds* of such involutions, according as the *fundamental points are real or imaginary*, and a *transition case* in which they coincide. The chief thing here for us, however, is the notion of involution itself; the distinction as to cases, i.e., the question as to the nature of the roots of the quadratic equation, is of secondary interest only.

These considerations, which can easily be carried over into three dimensions, of course, do not afford, indeed, an interpretation of the imaginary, but still they supply, insofar as configurations of order two are concerned, a *standpoint above the distinction between real and imaginary*. *Each configuration of second order is represented by a real polar system* and we can operate geometrically with this polar system as we can operate analytically with the real equations of the configuration.

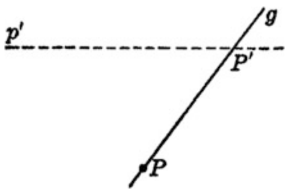

Figure 91

An *example* will show this more fully. Consider a curve of the second degree, i.e., of a polar system given in the plane, and consider also a straight line. This offers intuitively many possible cases according as the curve has or has not any real points whatever, and, if it has, whether the straight line cuts it in real points or not. In any case, the plane polar system will establish on the straight line g (see Fig. 91) a linear polar system, i.e., an involution. To every point P on g there corresponds in the first system a polar p', and this meets g in a point P'. The points (P, P') traverse the involution in question. We may enquire ex post about the fundamental points, and determine whether they are real or imaginary. In all this, we have put into geometric language just what we inferred from the equations in the beginning of this discussion.

We shall apply these considerations, in particular, to the *imaginary circular points and the imaginary spherical circle*. We said earlier that any two circles cut the line at infinity in the same two points, the imaginary circular points. This means now, geometrically, *that their polar systems set up on the line at infinity one and the same one-dimensional polar system*, *the same involution*. In fact, if we draw the tangents (see p. [57]) from an infinitely distant point P to a circle, then its polar p'_1, as the join of the points of tangency of these tangents from P, will be perpendicular to their common direction (see Fig. 92). Since all straight lines to the same point at infinity are parallel, the polar p'_2 of P, with respect to a second circle, will be perpendicular to the same direction as p'_1 and therefore parallel to p'_1. In other words, p'_1 and p'_2 meet the line at infinity in the same point P'. *Thus the polar systems of ail circles cut the line at infinity in one and the same polar system*, *the so-called* "*absolute involution*," whose *pairs of points*, *looked at from any finite point*, *appear in directions at right angles to each other*. **[132]**

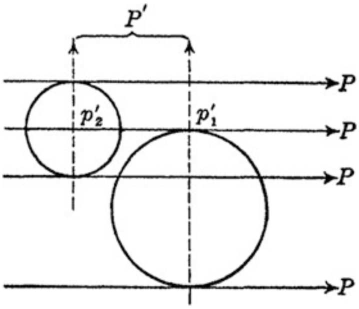

Figure 92

Let us now put these thoughts into analytic language. If we start from the homogeneous equation of the circle:

$$(\xi - a\tau)^2 + (\eta - b\tau)^2 - r^2\tau^2 = 0\,,$$

or

$$\xi^2 + \eta^2 - 2a\xi\tau - 2b\eta\tau + \left(a^2 + b^2 - r^2\right)\tau^2 = 0\,,$$

then the corresponding polar relation is

$$\xi\xi' + \eta\eta' - a(\xi\tau' + \xi'\tau) - b(\eta\tau' + \eta'\tau) + \left(a^2 + b^2 - r^2\right)\tau\tau' = 0\,.$$

From this we get the relation generated on the line at infinity if we put $\tau = \tau' = 0$:

$$\xi\xi' + \eta\eta' = 0\,, \quad \tau = 0\,, \quad \tau' = 0\,.$$

These equations are, in fact, independent of the special constants a, b, and r of the initial circle. Furthermore, it follows from analytic geometry that, because of the

first equation, two lines drawn to the points $(\xi, \eta, 0)$ and $(\xi', \eta', 0)$ are perpendicular to each other, so that we have actually obtained the theorem stated above.

Entirely analogous results hold for the *spheres of space*. They all generate on the plane at infinity one and the same, the so-called *absolute polar relation*, which is given by the equations

$$\xi\xi' + \eta\eta' + \zeta\zeta' = 0, \quad \tau = 0, \quad \tau' = 0.$$

Since the first equation says that the directions $\xi : \eta : \zeta$ and $\xi' : \eta' : \zeta'$ are
[133] perpendicular to each other, then *there corresponds to every point at infinity P that line at infinity, which is cut out by the plane perpendicular to the direction toward P from a finite point*. Thus we have a real geometric equivalent of the theorems concerning the imaginary spherical circle.

Staudt's Interpretation of Individual Imaginary Elements

It may be said, to be sure, that the imaginary is avoided rather than interpreted in this discussion. An actual interpretation of individual imaginary points, lines, and planes was first given by von Staudt in his "*Beiträge*" of 1856–60, by an extension of this theorem. I shall give this interpretation, also, because it is actually simple and ingenious; it seems strange and difficult only in von Staudt's abstract presentation. I shall follow the analytic presentation given by Otto Stolz in 1871.[56] Stolz and I were then together in Göttingen. He had read von Staudt, which I could never bring myself to do; hence I learned in personal intercourse with him not only these but many other interesting ideas of von Staudt with which I myself later worked a good deal. I wish here to give only the most important features of the train of thought, without carrying out the details fully. It will suffice if I confine myself to the plane.

Let us assume, to start with, *an imaginary point P*, given by its complex coordinates (ξ, η, τ). Let these be separated into their real and imaginary parts

$$\xi = \xi_1 + i\xi_2, \quad \eta = \eta_1 + i\eta_2, \quad \tau = \tau_1 + i\tau_2. \tag{1}$$

Now we wish to construct a *real figure* by means of which this point P can be interpreted, and the *connection is to be projective*, i.e., speaking more precisely, it is to remain unchanged under arbitrary real projective transformation.

1. The first necessary step for this is to fix attention upon the two real points P_1, P_2 whose homogeneous coordinates are, respectively, the real parts of the coordinates of P and the imaginary parts multiplied by $-i$:

$$P_1 : \xi_1, \eta_1, \tau_1; \quad P_2 : \xi_2, \eta_2, \tau_2. \tag{1a}$$

[56] *Die geometrische Bedeutung der complexen Elemente in der analytischen Geometrie*, Mathematische Annalen, vol. 4, p. 416, 1871.

These two points are different, i.e., the relation $\xi_1 : \eta_1 : \tau_1 = \xi_2 : \eta_2 : \tau_2$ is not valid, otherwise $\xi : \eta : \tau$ would behave like three real quantities and would represent therefore one real point. Hence P_1, P_2 determine a real straight line g whose equation is

$$\begin{vmatrix} \xi & \eta & \tau \\ \xi_1 & \eta_1 & \tau_1 \\ \xi_2 & \eta_2 & \tau_2 \end{vmatrix} = 0 . \tag{2}$$

This line contains the given imaginary point P, as well as the *conjugate imaginary* **[134]** *point* $\overline{P}$, whose coordinates are

$$\overline{\xi} = \xi_1 - i\xi_2 , \quad \overline{\eta} = \eta_1 - i\eta_2 , \quad \overline{\tau} = \tau_1 - i\tau_2 , \tag{$\overline{1}$}$$

since both coordinate triples (1), ($\overline{1}$) satisfy the equation of the straight line.

2. Of course the pair of points P_1, P_2, so constructed, can by no means pass as the representative of the imaginary point P, for they depend essentially upon the separate values of ξ, η, and τ, whereas, for the point P, it is only the ratios of these values which are characteristic. The same point P will therefore be represented if, instead of ξ, η, and τ, their products by an arbitrary complex constant $\rho = \rho_1 + i\rho_2$ are written in the form

$$\begin{cases} \rho\xi = \rho_1\xi_1 - \rho_2\xi_2 + i\,(\rho_2\xi_1 + \rho_1\xi_2) \; , \\ \rho\eta = \rho_1\eta_1 - \rho_2\eta_2 + i\,(\rho_2\eta_1 + \rho_1\eta_2) \; , \\ \rho\tau = \rho_1\tau_1 - \rho_2\tau_2 + i\,(\rho_2\tau_1 + \rho_1\tau_2) \; ; \end{cases} \tag{3}$$

but then we get, if we separate the real parts from the imaginary, instead of the points P_1, P_2, *other* real points P_1', P_2', whose coordinates are

$$\begin{cases} P_1': \quad \xi_1' : \eta_1' : \tau_1' = \rho_1\xi_1 - \rho_2\xi_2 : \rho_1\eta_1 - \rho_2\eta_2 : \rho_1\tau_1 - \rho_2\tau_2 \, , \\ P_2': \quad \xi_2' : \eta_2' : \tau_2' = \rho_2\xi_1 + \rho_1\xi_2 : \rho_2\eta_1 + \rho_1\eta_2 : \rho_2\tau_1 + \rho_1\tau_2 \, . \end{cases} \tag{3a}$$

If we consider the totality of pairs of points P_1' and P_2', given by all the values of ρ_1 and ρ_2, we have a geometric configuration in which only the ratios $\xi : \eta : \tau$ count, i.e., the "geometric" point P, which is therefore fitted to serve as representing P. Moreover, the connection with P is, in fact, projective, for, if we transform ξ, η, τ in any real linear manner, then ξ_1', η_1', τ_1', and ξ_2', η_2', τ_2' suffer the same substitution.

3. In order, now, to study the geometric nature of this totality of pairs of points, we note first that, whatever the value of ρ, the points P_1' and P_2' *lie on the line* P_1P_2 (see Fig. 93), since their coordinates obviously satisfy equation (2). Moreover, if we allow ρ to assume all complex values, i.e., ρ_1 and ρ_2 all real values (a common

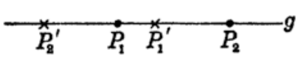

Figure 93

real factor makes no essential difference), then P_1' runs through all the real points of g, and P_2' represents always a second real point on g in unique correspondence with P_1'. Thus, for $\rho_1 = 1$, $\rho_2 = 0$, we have P_1 and P_2 as corresponding points. The correspondence stands out more clearly if we introduce the ratio

$$\frac{\rho_2}{\rho_1} = -\lambda\,.$$

[135] Then we have

$$\text{(3b)} \qquad \begin{cases} P_1': & \xi_1' : \eta_1' : \tau_1' = \xi_1 + \lambda\xi_2 : \eta_1 + \lambda\eta_2 : \tau_1 + \lambda\tau_2; \\ P_2': & \xi_2' : \eta_2' : \tau_2' = \xi_1 - \frac{1}{\lambda}\xi_2 : \eta_1 - \frac{1}{\lambda}\eta_2 : \tau_1 - \frac{1}{\lambda}\tau_2. \end{cases}$$

4. From these formulas we can infer also that, when λ varies, the points P_1 and P_2 become all the *point pairs of an involution on the straight line g*. For if we introduce a one-dimensional coordinate system on g, the homogeneous coordinates of the points P_1 and P_2 become linear integral functions of the parameters $\lambda_1' = \lambda$ and $\lambda_2' = -1/\lambda$, respectively, of the equations (3b). Hence the equation $\lambda_1' \cdot \lambda_2' = -1$ between the two parameters yields a symmetrical bilinear relation between the linear coordinates of P_1' and P_2', and consequently, in view of the definition on p. [131] (see also p. [119]), the assertion is proved.

5. The *fundamental points of this involution*, i.e., the points which correspond to each other, are given by $\lambda = -1/\lambda$, also by $\lambda = \pm i$. They are both imaginary, one being the point P with which we started, the other the conjugate imaginary $\overline{P}$. Thus far we have given only a new presentation of von Staudt's old theory. *Besides P we have considered the point $\overline{P}$, which, together with P, forms a one-dimensional configuration of the second degree, determined by a real quadratic equation, and we have then constructed the resulting involution as its real representative.* I remind you that such an involution is determined if we know *two of its point pairs*, say P_1, P_2 and P_1', P_2'. If this involution is to have imaginary fundamental points, it is necessary and sufficient that these point pairs should be *"in twisted position"*, i.e., that one of the points P_1' and P_2' should lie between P_1 and P_2 and the other outside of them.

6. In order to solve our problem completely, we need only a means for transforming the common representative of P and $\overline{P}$ into a *representative of P alone* (or of $\overline{P}$ alone). Von Staudt discovered such a means in 1856 as the result of a brilliant thought. The point P_1', with the coordinates $\xi_1 + \lambda\xi_2 : \eta_1 + \lambda\eta_2 : \tau_1 + \lambda\tau_2$ traverses, namely, the straight line g in a *perfectly definite direction* (see Fig. 94) if λ runs through all real values from 0 to $+\infty$ and back, through negative values to 0. It is easy to show that we should be led to just the same direction on g if we started

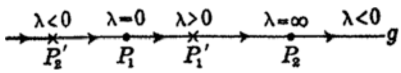

Figure 94

with the coordinates of P multiplied by an arbitrary ρ, i.e., if we considered the point $\xi_1' + \lambda\xi_2'$, ... Moreover, under real projective transformation of P, the direction of **[136]**
the arrow for the image point would follow from the one just determined, as a result of the same transformation. We shall, then, satisfy our requirements *if we make this arrow direction correspond*, *once for all*, *with the original point* P $(\xi_1 + i\xi_2, \ldots)$. Since the conjugate imaginary point $\overline{P}$ has the coordinates $\xi_1 + i(-\xi_2), \ldots$, we must, accordingly, assign as the direction of motion of P for positive increasing λ, the opposite of the direction just determined for the straight line g, thus achieving the desired distinction: *We distinguish between* $+i$ *and* $-i$ *simply by distinguishing between the positive and the negative passing of the real values of* λ.

Thus we have, at last, the following *rule for the construction of a unique and projectively invariant real geometric figure to represent the imaginary point* $\xi_1 + i\xi_2 : \eta_1 + i\eta_2 : \tau_1 + i\tau_2$: *Construct the points* $P_1(\xi_1 : \eta_1 : \tau_1)$ *and* $P_2(\xi_2 : \eta_2 : \tau_2)$, *their straight connecting line* g, *and that point involution* on g (or another point pair on g) *in which the points*

$$P_1'\,(\xi_1 + \lambda\xi_2 : \eta_1 + \lambda\eta_2 : \tau_1 + \lambda\tau_2) \quad \text{and} \quad P_2'\left(\xi_1 - \frac{1}{\lambda}\xi_2 : \eta_1 - \frac{1}{\lambda}\eta_2 : \tau_1 - \frac{1}{\lambda}\tau_2\right)$$

are always paired. Finally, *we add the arrow*, *giving it the direction in which* P_1' *moves with positive increasing* X.

7. It remains for us only to show *that*, *conversely*, *every such real figure*, *consisting of a straight line*, *two point pairs lying twisted on it* – P_1, P_2 *and* P_1', P_2' – (*or an involution range without real double points*), *together with a direction arrow*, *represents one and only one imaginary point.* I need not carry this proof out in detail. However, by choosing a suitable real constant factor, it is easy to attribute to the coordinates of P_2 such values ξ_2, η_2, τ_2 that the coordinates of P_1' and P_2' are proportional to

$$\xi_1 + \lambda\xi_2 : \eta_1 + \lambda\eta_2 : \tau_1 + \lambda\tau_2 \quad \text{and} \quad \xi_1 - \frac{1}{\lambda}\xi_2 : \eta_1 - \frac{1}{\lambda}\eta_2 : \tau_1 - \frac{1}{\lambda}\tau_2\,,$$

or, what is the same thing, that the double points of the assumed involution range have the coordinates $\xi_1 \pm i\xi_2, \ldots$ The sign of λ, which is thus far arbitrary, is to be chosen so that the direction of motion of the point $\xi_1 + \lambda\xi_2$, $\eta_1 + \lambda\eta_2$, $\tau_1 + \lambda\tau_2$ when λ increases positively from zero on, shall agree with the direction arrow. Then the point P, with coordinates $\xi_1 + \lambda\xi_2, \ldots$, in view of the preceding developments, will actually represent the given involution with the given direction of the arrow. Moreover, it can be shown, that we are led to the same coordinate ratios, i.e., to the same point P, if we start from another point pair of the involution.

The Positions of Imaginary Points and Straight Lines **[137]**

Having completed the discussion of our problem for the *point*, we can transfer the solution to the *straight line* in the plane by the principle of duality. *Accordingly*,

we have a real unique representation of a complex straight line by means of a real point, and two pairs of rays taken from the ray family by means of it and lying twisted (or *an involution of rays without real double rays*), *together with a definite direction of rotation in the family*. (See Fig. 95.)

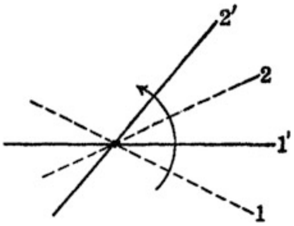

Figure 95

These results permit also the representation of all relations between complex and real elements, by means of tangible properties of real geometric figures. This fact constitutes the real value of these results. In order to make this clear by a concrete example, I shall show you the meaning, in this representation, of the statement *that a point P* (real or imaginary) *lies on a straight line g* (real or imaginary). Here we have, of course, to distinguish four cases:

1. Real point and real straight line.
2. Real point and imaginary straight line.
3. Imaginary point and real straight line.
4. Imaginary point and imaginary straight line.

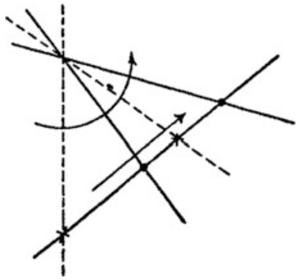

Figure 96

Case 1 needs no special explanation; it constitutes a fundamental relation of the usual geometry. In case 2, the given real point must lie also on the conjugate imaginary line; hence it *must be identical with the vertex of the ray family,* which we use to represent the imaginary line. Similarly, in case 3, the real line must be identical *with the range, which carries the point involution* that represents the given imaginary point. Case 4 is the most interesting. (See Fig. 96.) Obviously, in this case, the conjugate imaginary point must lie on the conjugate imaginary line, from

which it follows that each point pair of the involution range which represents P must lie on a pair of rays of the line involution which represents g, i.e., that *these two representing involutions must lie in perspective with each other*; moreover, it turns out that *the arrows of the two involutions are also in perspective*.

Summing up this discussion, we may say that we get – *in the general analytic geometry, which takes account also of the complex elements – a complete real picture of the plane, if we adjoin to the totality of the real points and straight lines of the plane, as new elements, the totality of given involution figures together with* **[138]** *the direction arrows*. It will suffice, perhaps, if I indicate in outline how we should construct this real image of complex geometry. In this I shall follow the order in which the basic theorems of elementary geometry are now usually presented.

1. We start with the *existence theorems,* which take accurate account of the presence of the elements just indicated of a domain extended with regard to ordinary geometry.

2. Then follow the *theorems of connection*, which state that, also in the extended domain defined in 1, *through two points there goes one and only one straight line* and that *two straight lines have one and only one common point*. There are four cases to be distinguished here, just as above, according to the reality of the given elements, and it is interesting to determine in what real constructions with point and line involutions these complex relations find their image.

3. As to the *laws of order*, there arises here, in contrast with real relations, an *entirely new situation*. In particular, the totality of real and complex points on a fixed straight line constitute a *two-dimensional continuum*, as do also all the rays through a fixed point. Everyone, indeed, is accustomed, from the theory of functions of a complex variable, to represent the aggregate of values of a complex variable by all the points of a plane.

4. Concerning the *theorems of continuity*, I shall only point out how we represent the complex points, which lie arbitrarily near a real point. For this purpose, we draw a real straight line through the real point P (or through a neighbouring real point) and we take upon it two such point pairs P_1, P_2 and and P_1', P_2' in twisted position (see Fig. 97) such that two points P_1, P_1' of different pairs lie close to each other and to P. If we now let P_1 and P_1' move into coincidence, the involution determined by these pairs degenerates, i.e., the two double points which were complex coincide with $P_1 = P_1'$. Each of the two imaginary points represented by the involution (with the one or the other arrow) thus transforms continuously into a point near P or, indeed, into P itself. We must, of course, work our way carefully into these representations of continuity in order to use them with profit.

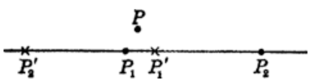

Figure 97

If this entire construction is prolix and bothersome, in comparison with the ordinary real geometry, it can, on the other hand, supply incomparably more.

[139] In particular, it can raise algebraic configurations, as a totality of their real and complex elements, to complete geometric intuitiveness. With it, we can make geometrically intuitive, by means of the figures, such theorems as the *fundamental theorem of algebra*, or the *theorem of Bézout* that two curves of degrees m and n have, in general, $m \cdot n$ common points. To achieve this, we should have to work out the theorems in a much more detailed manner to become intuitive than has yet been done. However, all the essential material for such an investigation can already be found in the literature.

In most cases, to be sure, the application of this geometric interpretation, notwithstanding its theoretical advantages, might create such complications that we should be satisfied with its theoretical possibilities and return actually to the more naive standpoint: a complex point is the aggregate of complex coordinate values with which, to a certain extent, one can operate as with real points. As a matter of fact, this use of imaginary elements, in complete disregard of all questions of theory, has always proved fruitful in dealing with the imaginary circular points and the imaginary spherical circle. As we saw, Poncelet was the first to use the imaginary in this sense. Other French geometers followed, notably Michel Chasles and Gaston Darboux. In Germany, this conception of the imaginary was used particularly by Lie with great success.

With this digression on the imaginary, I bring to a close the second main division of this course and turn to a new chapter.

Third Part Systematic Discussion of Geometry and Its Foundations [140]

I. The Systematic Discussion

In this chapter, we shall at first use geometric transformations to achieve a systematic division of the entire field of geometry, one which will enable us, from *one* standpoint, to overlook the separate parts and their interrelations.

1. Survey of the Structure of Geometry

Group Theory as a Principle to Systematise Geometry

We are concerned here with ideas such as those that I developed systematically in my *Erlanger Programm*[57] of 1872. You will find information as to the development of these ideas since that time in the *Enzyclopädie* report by Gino Fano: *Die Gruppentheorie als geometrisches Einteilungsprinzip* (Enz. III A.B. 4b).

1. As in the past, we shall consistently make use of analysis to gain mastery of geometric relations by *thinking of the totality of points in space as represented by the totality of values of the three "coordinates" x, y, and z.* To every transformation of space there corresponds, then, a certain transformation of these coordinates. From the beginning of our discussions we have recognised four kinds of transformations of particular significance, which are represented by certain *special linear substitutions of x, y,* and *z: Parallel displacements*, *rotations about the origin O*, *reflections in O*, *and similarity transformations with O as centre*.

2. It might be supposed that the introduction of coordinates would bring about complete identity between analysis of three independent variables (x, y, z) and geometry in a specific sense. Such is not the case, however. As I have already emphasised (p. [25]–[26]), geometry is concerned only with those relations between the coordinates which remain unchanged by the linear substitutions mentioned in 1, **[141]** regardless of whether these are thought of as changes in the system of coordinates or as transformations of space. *Thus geometry is the invariant theory of those linear substitutions*. All non-invariant equations between coordinates, on the other hand,

[57] *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen, 1872. Reprinted in *Mathematische Annalen*, vol. 43, pp. 63 sqq., 1893; and F. Klein, *Gesammelte mathematische Abhandlungen*, vol. 1, pp. 460 sqq., Berlin, Springer, 1921.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_12

e.g., the statement that a point has the coordinates $(2, 5, 3)$, have reference only to a definite coordinate system, fixed once for all. Such a discussion would belong to a science which must individualise each point for itself and consider its properties separately: to *topography*, or, if one prefers, *geography*. As an aid to understanding, I call to your attention several *examples of geometric properties*: The statement that two points are separated by a *distance*, when once a unit of length is chosen, means for us in the present conception that we can construct from their coordinates (x_1, y_1, z_1) and $(x_2,\ y_2,\ z_2)$ an expression $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ which remains unchanged under all those linear substitutions, or is multiplied by a factor that is independent of the special location of the points. A similar meaning must be given to the statements that two straight lines are inclined at a certain *angle*, that a conic section has certain *principal axes* and *foci*, etc.

The totality of these geometric properties we shall call *metric geometry*, in order to distinguish it from other *kinds of geometry*. We shall obtain the latter by separating out, for specific consideration, according to a definite principle, certain groups of theorems of metric geometry. Accordingly, all these newer kinds of geometry are, at least for the immediate purpose, parts of metric geometry as the most inclusive "kind of geometry."

3. We start with the *affine transformations*, which we have studied carefully earlier, i.e., with the integer linear substitutions in x, y, and z:

$$\begin{cases} x' = a_1 x + b_1 y + c_1 z + d_1\,, \\ y' = a_2 x + b_2 y + c_2 z + d_2\,, \\ z' = a_3 x + b_3 y + c_3 z + d_3\,. \end{cases}$$

Under this transformation all the transformations mentioned in 1 are embraced as special cases, and we select from among the totality of geometric concepts and theorems the narrower group of those, which remain unchanged under all affine transformations. This aggregate of concepts and theorems we consider as the first new kind of geometry, the so-called *affine geometry* or the *invariant theory of affine transformations*.

From the knowledge we have acquired of affine transformations, we can select, at once, the concepts and the theorems of this geometry. I recall here only a few: In affine geometry, distance and angle are not meaningful concepts. Likewise, the
[142] notion of principal axes of a conic section, and the distinction between circle and ellipse become blurred. There remains, however, the distinction between *finite* and *infinite* space and everything, which depends upon it, such as the notion of *parallelism of two straight lines*, the division of conic sections into *ellipses*, *hyperbolas*, *parabolas*, etc. Moreover, the notions of *centre* and *diameter of a conic section*, and particularly the relation of *conjugate* diameters, remain.

4. We shall now proceed to *projective changes*, i.e., we shall introduce the linear fractional transformations

$$x' = (a_1 x + b_1 y + c_1 z + d_1) : (a_4 x + b_4 y + c_4 z + d_4)$$
$$y' = (a_2 x + b_2 y + c_2 z + d_2) : (a_4 x + b_4 y + c_4 z + d_4)$$
$$z' = (a_3 x + b_3 y + c_3 z + d_3) : (a_4 x + b_4 y + c_4 z + d_4)$$

which include the affine transformations as special cases. Geometric properties that remain unchanged under these transformations must certainly belong also to affine geometry. Thus, from affine geometry, we separate out the so-called *projective geometry* as the *invariant theory of projective transformations*. The step-by-step sifting of affine and projective geometry from metric geometry can be compared to the procedure of the chemist, who, by applying ever-stronger reagents, isolates increasingly valuable ingredients from his compound. Our reagents are first affine transformations, and then projective transformations.

As to the theorems of projective geometry, it should be emphasised that the exceptional role of infinity and the concepts connected with it in affine geometry all now have fallen away. There is only *one kind of proper conic section*. There still remains, however, for example, the *relation between pole and polar*, and likewise the *generation of the conic section by means of projective families*, which we discussed earlier (pp. [104]–[105]).

By means of the same principle, we may also now pass *from metric geometry to other kinds of geometry*. One of the most important is the geometry of reciprocal radii.

5. *The geometry of reciprocal radii*. This comprises the aggregate of those theorems of metric geometry, which retain their validity under all *transformations of reciprocal radii*. In this geometry, the concepts of straight line or plane have no independent meaning; they appear as special cases in the notion of *circle or sphere*, respectively.

6. Finally, let me propose still another kind of geometry, which, in a sense, is obtained by the most careful sifting process of all, and which, therefore, includes the fewest theorems. This is *analysis situs*, which I mentioned earlier (pp. [113] sqq.). Here one is concerned with the *aggregate of properties*, *which persist under all transformations which are biunique and continuous*. In order to avoid assigning an exceptional place to infinity, which would go into itself in all such transforma- **[143]** tions, we can adjoin either the *projective transformations*, or the *transformations by means of reciprocal radii*.

We shall define still more sharply the scheme thus outlined, by introducing the *concept of a group*. As we have already seen, *an aggregate of transformations is called a group if the combination of two of its transformations gives again a transformation of the totality*, *and if the inverse of every transformation also belongs to the totality*. Examples of groups are the totality of *all movements*, or that of *all collineations* (projective transformations); for two motions combine into a motion, two collineations into a collineation, and in both cases there exists an inverse to every transformation.

If we look back at our different kinds of geometry, we see that the transformations, which play a role in each case always form a group. In the first place, all linear substitutions, which leave unchanged the relations of *metric geometry* – displacements, rotations, reflections, similarity transformations – obviously form a group, which one calls the *principal group of the transformations of space*. It is easy to establish the analogous significance of the *affine group* of all affine transformations for *affine geometry*, and of the *projective group* of all collineations for *projective geometry*. The theorems of the *geometry of reciprocal radii* remain valid under all transformations that are obtained by combining any reciprocal radii transformations with substitutions of the principal group. All these again form a group, namely, that of *reciprocal radii*. For *analysis situs*, finally, one has to do with the *group of all continuous biunique distortions*.

We wish now to determine upon how many independent parameters a single operation in each of these groups depends. In the principal group, the motions involve six parameters, to which one must add one parameter for the change in unit length, so that altogether there are *seven parameters*. We express this by calling the *principal group a* G_7. The equations of the general affine transformation contain $3 \cdot 4 = 12$ arbitrary coefficients; those of the projective $4 \cdot 4 = 16$, whereby a factor common to all is unessential. It follows that the *affine group is a* G_{12}, and that the *projective group is a* G_{15}. The *group of the reciprocal radii* turns out to be a G_{10}. Finally, the *group of all continuous distortions* has no finite number of parameters whatever; the operations of this group depend, rather, upon arbitrary functions, or,
[144] if one wishes, upon infinitely many parameters. We may say that it is a G_∞.

In the connection between different kinds of geometry and groups of transformations, which we have just discussed, there appears a *fundamental principle,* which can serve to characterise all possible geometries. It was just this which constituted the leading thought of my *Erlanger Programm*: *Given any group of transformations in space, which includes the principal group as a subgroup, then the invariant theory of this group gives a definite kind of geometry, and every possible geometry can be obtained in this way.* Thus each geometry is characterised by its group, which, therefore, assumes the leading place in our considerations.

This principle has been completely carried through in the literature only for the first three cases of our outline. We shall devote some time to these as the most important or the best known, and we shall pay special attention to the passage from one of them to the other.

We shall adopt an order opposite to that just followed, and start with projective geometry, that is, with the G_{15} *of all projective transformations*, which we may write in the homogeneous form

$$
\text{(1)} \qquad \left\{
\begin{aligned}
\rho'\xi' &= a_1\xi + b_1\eta + c_1\zeta + d_1\tau\,,\\
\rho'\eta' &= a_2\xi + b_2\eta + c_2\zeta + d_2\tau\,,\\
\rho'\zeta' &= a_3\xi + b_3\eta + c_3\zeta + d_3\tau\,,\\
\rho'\tau' &= a_4\xi + b_4\eta + c_4\zeta + d_4\tau\,.
\end{aligned}
\right.
$$

In order to get from this to the affine group, we begin with the remark that a projectivity is an affine transformation if it sends the plane at infinity into itself, i.e., if to every point with vanishing τ there corresponds a point with vanishing τ'. This will happen if $a_4 = b_4 = c_4 = 0$; hence, if we divide each of the equations (1) by $\rho'\tau'$ in order to get non-homogeneous equations, and replace $a_1 : d_4, \ldots$ simply *by* $a_1, \ldots$, we obtain

$$(2) \qquad \begin{cases} x' = a_1 x + b_1 y + c_1 z + d_1\,, \\ y' = a_2 x + b_2 y + c_2 z + d_2\,, \\ z' = a_3 x + b_3 y + c_3 z + d_3\,. \end{cases}$$

These are, in fact, the old affine formulas: *The condition that the plane at infinity shall remain unchanged separates out of the projective G_{15} a twelve-parameter "subgroup", namely, the affine group.*

Similarly, we obtain the *principal group G_7 by selecting out the projectivities* (*or the affine transformations*) which leave invariant *not only the plane at infinity but also the imaginary spherical circle*, i.e., under which, to every point which satisfies the equations $\xi^2 + \eta^2 + \zeta^2 = 0$ and $\tau = 0$, there corresponds a point which satisfies **[145]** the same equations. This assertion is easily verified. You need only bear in mind that our condition fixes, to within a constant factor, the six (homogeneous) constants of a conic section, which corresponds to the imaginary spherical circle by virtue of an affine transformation in the plane $\tau' = 0$. Hence it imposes upon the twelve constants of the affine transformation $6 - 1 = 5$ conditions, so that precisely the $12 - 5 = 7$ parameters of the G_7 remain.

Cayley's Principle: Projective Geometry is All Geometry

This whole manner of viewing the subject was given an important turn by the great English geometer Arthur Cayley[58] in 1859. Whereas, up to this time, it had seemed that affine and projective geometry were poorer sections of metric geometry, Cayley made it possible, on the contrary, *to look upon affine geometry as well as metric geometry as special cases of projective geometry*, "*projective geometry is all geometry*." This apparently paradoxical connection arises from the fact that one adjoins to the figures under investigation certain configurations, namely, the *plane at infinity*, or, as the case may be, the imaginary *spherical circle* which lies in it; hence *the affine or the metric properties*, *respectively*, *of a figure are nothing but the projective properties of the figure thus extended.*

Let me illustrate this by two very simple examples, in which I shall present well-known facts in a somewhat altered form. The statement that two straight lines are parallel has no immediate meaning in projective geometry. However, if we add

[58] In *A sixth memoir upon quantics*, Philosophical Transactions of the Royal Society of London, 1859 = *Collected Mathematical Papers*, vol. 2 (Cambridge, 1889), pp. 561–592.

the plane at infinity to the given configuration (the two straight lines), we have the purely projective statement (see p. [98]) that two given straight lines intersect on a given plane. We have a similar situation if a straight line is perpendicular to a plane. We can resolve this (see pp. [132]–[133]) into a polar relation (a projective property) of the given figure extended by the addition of the imaginary spherical circle (see Fig. 98): The point trace P_∞ of the straight line and the line trace g_∞ of the plane, in the plane at infinity, are pole and polar with respect to the imaginary spherical circle.

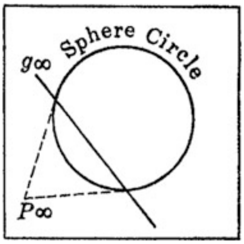

Figure 98

I should like to carry out more fully the line of thought, which I have indicated briefly here and show how it leads to a *completely systematic teaching structure of geometry*. The greatest credit for this belongs to the English mathematicians. I have already mentioned Cayley. Next to him I should place J. J. Sylvester and George Salmon of Dublin. These men, *beginning in* 1850, created the algebraic discipline **[146]** which is called, in a narrower sense, *the invariant theory of linear homogeneous substitutions*,[59] and which, under the guidance of Cayley's principle, makes possible a complete *systematic structure of geometry on an analytic basis*. In order to understand this system, it will be necessary for us to devote a little time to the theory of invariants.

2. Digression on the Invariant Theory of Linear Substitutions

The Systematics of Invariant Theory

Of course, I shall be able to present only the main results and lines of thought, without going into details and proofs. As to the *literature* of this wide field, I refer you, above all; to the report by W. Franz Meyer: *Die Fortschritte der projectiven Invariantentheorie im letzten Vierteljahrhundert* in the first volume of *Jahresberichte der deutschen Mathematiker-Vereinigung* (1892), as well as to the report on *Invari-*

[59] The words "invariant theory" are used also in a wider sense with reference to arbitrary transformation groups. In the narrower sense, as we shall use them in these pages, they were first applied by Sylvester.

antentheorie in the *Enzyklopädie* by the same author (Enzyklopädie vol. I B 2). All that is needed in the geometry of invariant theory especially is to be found in the textbooks of G. Salmon,[60] which have contributed most to spread the ideas, which arise here. The German edition of Salmon's book by Wilhelm Fiedler has always enjoyed an unusually wide use. The lectures of Alfred Clebsch,[61] which Ferdinand Lindemann edited, are in the same category.

1. Going over now to our subject, let us think of any number of given variables, and let us speak, accordingly, of a *binary*, *ternary*, *quaternary*, … *region*. To enable us to consider the variables in the first three cases ultimately as homogeneous coordinates in a straight line, a plane, or in space, we designate them by the symbols

$$\xi, \tau; \quad \xi, \eta, \tau; \quad \xi, \eta, \zeta, \tau,$$

where $\tau = 0$ will always characterise the infinitely distant elements.

2. We consider the *groups of all homogeneous linear substitutions of these variables*. At present we shall have in mind not merely the ratios of the variables, as will be the case later in projective geometry, but also their individual values. We **[147]** may write these substitutions in the form

$$\begin{aligned} \xi' &= a_1\xi + d_1\tau\,, \\ \tau' &= a_4\xi + d_4\tau\,; \end{aligned} \qquad \begin{aligned} \xi' &= a_1\xi + b_1\eta + d_1\tau\,, \\ \eta' &= a_2\xi + b_2\eta + d_2\tau\,, \\ \tau' &= a_4\xi + b_4\eta + d_4\tau\,; \end{aligned} \qquad \begin{aligned} \xi' &= a_1\xi + b_1\eta + c_1\zeta + d_1\tau\,, \\ \eta' &= a_2\xi + b_2\eta + c_2\zeta + d_2\tau\,, \\ \zeta' &= a_3\xi + b_3\eta + c_3\zeta + d_3\tau\,, \\ \tau' &= a_4\xi + b_4\eta + c_4\zeta + d_4\tau\,. \end{aligned}$$

The number of parameters in these three groups is 4, 9, and 16, respectively. For convenience, we shall use habitually in the formulas only the variables ξ and τ, and we shall write out only the terms involving these two, with dots between them. If we are dealing then with the binary region, we simply ignore these dots; for the ternary and quaternary regions, we replace the dots by terms in η, or in η and ζ, analogous to terms already written out. In general, then, we speak of the variables $\xi, \ldots, \tau$ and of the linear substitutions in them

$$(1) \qquad \left\{ \begin{aligned} &\xi' = a_1\xi + \cdots + d_1\tau\,, \\ &\cdots\cdots\cdots \\ &\tau' = a_4\xi + \cdots + d_4\tau\,. \end{aligned} \right.$$

3. As to the *objects of the invariant theory*, we shall consider the question in two different levels. In the first level we think of *any individual systems of values of the variables* $\xi_1, \ldots, \tau_1$; $\xi_2, \ldots, \tau_2$, $\xi_3, \ldots, \tau_3$; …, which, in the spirit of geometry, we may designate outright as *points* $1, 2, 3, \ldots$ Each of these systems of values is

[60] *Analytic Geometry* I. *Conic Sections*; II. *Higher Plane Curves*; III. *Space*; IV. *Lectures on the Algebra of Linear Transformations*. German by W. Fiedler, Leipzig (Teubner). Each volume in several editions. [Vol. I newly edited by F. Dingeldey; III by K. Kommerell and A. Brill.]

[61] *Vorlesungen über Geometrie*, edited by F. Lindemann, Leipzig (Teubner), 1st ed., 1876 et seq., 2nd ed., 1906 et seq.

subjected to the substitutions of the group (1), and we are concerned with *setting up combinations of our systems of values, which remain invariant under these simultaneous substitutions*.

4. The second level of the problem considers, in addition to such points, also *functions of the variables*, and, primarily, *rational integer functions*. We may confine ourselves, indeed, to *homogeneous* rational integer functions (called *forms* by the invariant theory), since the terms of like dimension substitute as such themselves, anyway, by reason of the homogeneity of the substitutions. Thus we shall consider the *linear forms*

$$\phi = \alpha\xi + \cdots + \delta\tau$$

the quadratic forms

$$f = A\xi^2 + \cdots + 2G\xi\tau + \cdots + K\tau^2$$

[148] and so on. We can also examine simultaneously *several forms* of like dimension, in which case we distinguish them by indices, e.g.,

$$\phi_1 = \alpha_1\xi + \cdots + \delta_1\tau\,; \quad \phi_2 = \alpha_2\xi + \cdots + \delta_2\tau\,; \quad \ldots$$

Similarly, we could start with *forms in several series of variables*, e.g., with the bilinear forms

$$f = \mathsf{A}\xi_1\xi_2 + \cdots + \varDelta\xi_1\tau_2 + \cdots + N\tau_1\xi_2 + \cdots + \varPi\tau_1\tau_2\,.$$

In order to make clear the general problem, which arises here, we must first inquire *how the coefficients of these forms are transformed* when we subject the variables to the substitutions of the group (1) and prescribe that the value of the form ϕ or f shall remain unchanged. Considering first the linear form, let us place

$$\phi = \alpha\xi + \cdots + \delta\tau = \alpha'\xi' + \cdots + \delta'\tau'\,.$$

If we introduce for $\xi', \ldots, \tau'$ the expressions (1), we get, in the variables $\xi, \ldots, \tau$, the identities

$$\begin{aligned}\alpha\xi + \cdots + \delta\tau &= \alpha'(a_1\xi + \cdots + d_1\tau) + \cdots + \delta'(a_4\xi + \cdots + d_4\tau)\\ &= (\alpha' a_1 + \cdots + \delta' a_4)\xi + \cdots + (\alpha' d_1 + \cdots + \delta' d_4)\tau\,,\end{aligned}$$

from which we obtain

$$(2) \qquad \left\{\begin{array}{l}\alpha = a_1\alpha' + \cdots + a_4\delta'\,,\\ \cdots\cdots\cdots\\ \delta = d_1\alpha' + \cdots + d_4\delta'\,.\end{array}\right.$$

Thus the new coefficients $\alpha', \ldots, \delta'$ of the linear form are connected with the old $\alpha, \ldots, \delta$ *by another linear substitution*, which is related in a simple way to (1): the

vertical and the horizontal rows in the array of coefficients are interchanged (the substitution is “transposed”) and, furthermore, the places of the old (unaccented) and the new (accented) magnitudes are interchanged. This new substitution (2) is called *contragredient* to the original substitution (1) and we say, briefly, that the coefficients $\alpha, \ldots, \delta$ of a linear form substitute themselves contragredient to the variables $\xi, \ldots, \tau$. The series of variables $\xi_1, \ldots, \tau_1$; $\xi_2, \ldots, \tau_2$; ... which are all subjected to the same substitution (1), are called, in analogous terminology, *cogredient variables*.

Going over now to the *quadratic form f*, let us inquire first how the quadratic terms $\xi^2, \ldots, \xi\tau, \ldots, \tau^2$ entering there behave under the linear substitution (1). From (1), we find at once, for the quadratic terms of the new variables, the formulas

$$(3) \qquad \begin{cases} \xi'^2 = a_1^2\xi^2 + \cdots + 2a_1d_1\xi\tau + \cdots + d_1^2\tau^2\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \xi'\tau' = a_1a_4\xi^2 + \cdots + (a_1d_4 + a_4d_1)\,\xi\tau + \cdots + d_1d_4\tau^2\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \tau'^2 = a_4^2\xi^2 + \cdots + 2a_4d_4\xi\tau + \cdots + d_4^2\tau^2\,. \end{cases}$$

We can express these relations briefly as follows. The quadratic terms of the vari- **[149]**
ables undergo, simultaneously with the variables themselves, a homogeneous linear substitution which can be derived immediately from (1). Since *f* is a *linear* form in these quadratic terms, we infer, by repetition of the foregoing reasoning, that the *coefficients* $A, \ldots, 2G, \ldots, K$ *undergo a transformation which is linear and homogeneous, and which is, indeed, contragredient to the substitution* (3) *of the terms* $\xi^2, \ldots, \xi\tau, \ldots, \tau^2$; i.e., the equations between $A, \ldots, 2G, \ldots, K$ and $A', \ldots, 2G', \ldots, K'$ are obtained from (3) just as (2) are from (1).

5. We can now formulate the *general problem of the theory of invariants*. Given any set of points $1, 2, \ldots$, and also certain linear, quadratic, or even higher forms $\phi_1, \phi_2, \ldots, f_1, f_2, \ldots$, then we mean by an *invariant* a function of the coordinates $\xi_1, \ldots, \tau_1$; $\xi_2, \ldots, \tau_2, \ldots$, and of the coefficients $\alpha_1, \ldots, \delta_1$; $\alpha_2, \ldots, \delta_2$; ...; $A_1, \ldots, K_1$; $A_2, \ldots, K_2$; ..., which remains unchanged under the linear substitutions (1) of the variables and the associated substitutions of the systems of coefficients, which we have just determined. *The aggregate of all whatever possible invariants is to be studied.*

The words *covariant* and *contravariant* are used sometimes for particular kinds of what are designated above in general as invariants. If the series of variables $\xi_1, \ldots, \tau_1$; $\xi_2, \ldots, \tau_2$, ... themselves occur in the invariant expression, we speak of *covariants*, and if coefficients of linear forms $\alpha_1, \ldots, \delta_1$; $\alpha_2, \ldots, \delta_2$; ... appear in it, we say *contravariant*. The word *invariant* is then confined to the expressions which contain neither such coordinates $\xi_1, \ldots$ nor coefficients $\alpha_1, \ldots$, but are made up only of coefficients of quadratic or higher forms. The reason why these two cases are emphasised and contrasted is that the series of variables $\xi, \ldots, \tau$ on the one hand, and $\alpha, \ldots, \delta$ on the other, show a certain reciprocal behaviour: if one of them undergoes a linear substitution, the other experiences the contragredient substitution, no matter with which series we start. Hence we can derive from every

invariant expression of the one sort, by suitable rearrangement, a similar one of the other sort. As for the geometric interpretation, we have here obviously an expression of the *principle of duality*, for $\alpha, \ldots, \delta$ become homogeneous straight line or plane coordinates if we think of $\xi, \ldots, \tau$ as point coordinates. However, the distinction as to whether or not $\xi, \ldots, \tau$, or $\alpha, \ldots, \delta$, actually appear in the expressions in question has, of course, no fundamental significance. We shall, in general, from now on, use the word *invariant* in the more comprehensive sense.

6. We shall now conceive of the notion of invariant more sharply in another di-
[150] rection, so as to make it possible to build up the theory in an orderly way. From now on, we shall think of invariants only as *rational functions of the coordinates and the coefficients* and which, moreover, are *homogeneous* in the coordinates of every point and in the coefficients of every form that occurs. We can express each such rational function as the *quotient of two rational integer homogeneous functions*, and we shall investigate these by themselves. Since a factor common to numerator and denominator does not alter the value of the quotient, these terms need not be invariants, in the sense thus far used, but may possibly be multiplied by a certain factor under each linear substitution.

It can be shown that this factor depends only on the coefficients of the substitution, and that it is necessarily a *power of the determinant of the substitution*:

$$r = \begin{vmatrix} a_1 \cdots d_1 \\ \cdots\cdots \\ a_4 \cdots d_4 \end{vmatrix} .$$

We come thus finally to the *consideration of those rational integer homogeneous functions of the given series of quantities,* which, under linear substitution of the *variables and the coefficients* (*as we have set them up*) *are multiplied by a power* r^λ *of the determinant of the substitution*. These we call *relative invariants*, since the changes they undergo are always unessential and they remain entirely unchanged under all substitutions for which $r = 1$. The exponent λ is called the *weight of the invariant*. By contrast, we call that which we have heretofore designated as invariant an *absolute invariant*. *Thus every absolute invariant is the quotient of two relative invariants of the same weight.*

7. With this we have actually gained a *point of view for the systematisation of the theory of invariants*. The simplest relative invariants will be polynomials of the *lowest possible degree* in the given series of variables. Starting with them, we should ascend to those of higher degree. If j_1, j_2 are any two relative invariants, then every product of their powers $j_1^{\kappa_1} \cdot j_2^{\kappa_2}$ will also be a relative invariant. For, if the substitution brings to j_1 the factor r^{λ_1} and to j_2 the factor r^{λ_2} then $j_1^{\kappa_1} \cdot j_2^{\kappa_2}$ will reproduce itself except for the factor $r^{\kappa_1\lambda_1+\kappa_2\lambda_2}$. If we now construct a sum of such terms, each multiplied by a constant factor

$$\sum_{(\kappa_1,\kappa_2,\ldots)} c_{\kappa_1,\kappa_2,\ldots} j_1^{\kappa_1} j_2^{\kappa_2} \cdots ,$$

and if we make sure that the individual summands are all multiplied by the same power of r, i.e., that they all have the same weight (are "isobaric"), then we have again, obviously, a relative invariant of higher degree, since the factor of the individual terms can be placed before the summation sign.

The *central problem* of the theory of invariants is, naturally, the question as to whether or not we can always get *all* the invariants in this way. *What is, in each given case, the complete system of lowest invariants from which one can build up, rationally and integrally, in the way indicated, all relative invariants?* The principal theorem, however, is that *to every finite number of given quantities there is always a finite "complete invariant system," i.e., a finite number of invariants from which all others can be built up rationally and integrally.* The credit for these definitive results in the systematic theory of invariants goes to the German researchers Paul Gordan and David Hilbert. The memoir by Hilbert in volume 36 of the *Mathematische Annalen*[62] is especially noteworthy. **[151]**

Simple Examples

I shall now take up some *simple examples*, such as we shall use afterward in geometry, in order to explain somewhat more the abstract development, which we have been considering. Here, of course, I shall give outlines rather than proofs.

1. Let us assume, first, that we have merely a *number of points in a binary region*:

$$\xi_1, \tau_1\,; \quad \xi_2, \tau_2\,; \quad \xi_3, \tau_3\,; \ldots$$

Here we have the interesting theorem that *the simplest invariants are furnished by the two-rowed determinants which can be formed from these coordinates, and that these determinants constitute the complete invariant system.* With two points 1 and 2, we can set up a two-rowed determinant

$$\Delta_{12} = \begin{vmatrix} \xi_1 & \tau_1 \\ \xi_2 & \tau_2 \end{vmatrix}.$$

This is actually a rational integer function of the variables, and is also homogeneous both in (ξ_1, τ_1) and (ξ_2, τ_2). We recognise the invariant nature of this determinant at once if we apply the rule for multiplying determinants to the calculation:

$$\Delta'_{12} = \begin{vmatrix} \xi'_1 \tau'_1 \\ \xi'_2 \tau'_2 \end{vmatrix} = \begin{vmatrix} a_1\xi_1 + d_1\tau_1, & a_4\xi_1 + d_4\tau_1 \\ a_1\xi_2 + d_1\tau_2 & a_4\xi_2 + d_4\tau_2 \end{vmatrix} = \begin{vmatrix} a_1 d_1 \\ a_4 d_4 \end{vmatrix} \cdot \begin{vmatrix} \xi_1\tau_1 \\ \xi_2\tau_2 \end{vmatrix} = r \cdot \Delta_{12}$$

Thus the invariant has the *weight* 1.

[62] *Über die Theorie der algebraischen Formen*, vol. 36, pp. 473 sqq., 1890.

In the same way, n points $1, 2, \ldots, n$ have altogether $n(n-1)/2$ invariants of weight 1:

$$\Delta_{ik} = \begin{vmatrix} \xi_i \tau_i \\ \xi_k \tau_k \end{vmatrix} \quad (i, k = 1, 2, \ldots, n)$$

To prove that these determinants constitute the *complete* invariant system, i.e., that *every relative invariant of the n points can be expressed as a sum of isobaric terms*:

$$\sum C \cdot \Delta_{ik}^{s} \Delta_{lm}^{l} \ldots$$

[152] would take us too far. We obtain the most general rational absolute invariants from the relative invariants, as quotients, where numerator and denominator are of equal weight; thus a simple example of an absolute invariant would be the quotient Δ_{ik}/Δ_{lm}.

In connection with this example, I should like to explain a finer concept formation, which plays an important role in the theory, namely that of the *syzygy* (i.e., a coupling together, or connecting, of invariants). It can happen, namely, *that certain of those aggregates of the fundamental invariants vanish*. Thus we have, for example, with four points

$$\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{42} + \Delta_{14}\Delta_{23} = 0\,.$$

This amounts to nothing more than a known determinant identity, which we have used, in fact, on occasion (see p. [33]). Such an identity between invariants of the complete system is called a *syzygy*. If we have several such syzygies, we can form new ones from them by multiplication and addition, and we may ask, as with the invariants themselves, concerning the *complete system of syzygies*, out of which all the others can be formed in this way. The theory shows *that there is always a finite system of this sort*. In the case of four points, for example, this complete system consists of the single equation above, i.e., all identities existing between the six determinants $\Delta_{12}, \ldots, \Delta_{34}$ are consequences of that one. In the case of five or more points, the complete system consists of all the equations of this type. Knowledge of these syzygies is, of course, of fundamental importance for the knowledge of the whole invariant system; for, if two isobaric aggregates of the simplest invariants differ by terms which have as a factor the left side of a syzygy, they are identical and do not need to be counted twice.

2. Similarly, *if we have single points in a ternary or quaternary region*, then the full invariant system consists, in precisely the same way, of the *three-rowed or four-rowed determinants formed from the coordinates*. In the ternary region, for example, the fundamental invariant of three points is again of weight 1:

$$\Delta_{123} = \begin{vmatrix} \xi_1 & \eta_1 & \tau_1 \\ \xi_2 & \eta_2 & \tau_2 \\ \xi_3 & \eta_3 & \tau_3 \end{vmatrix}\,.$$

I shall leave to you all the rest of the details; in particular, how the syzygies are set up here.

3. Let us now proceed to consider a *quadratic form*, in, say, a quaternary region:

$$f = A\xi^2 + 2B\xi\eta + C\eta^2 + 2D\xi\zeta + 2E\eta\zeta + F\zeta^2 + 2G\xi\tau + 2H\eta\tau + 2J\zeta\tau + K\tau^2 .$$

We can write down at once *one invariant* which depends only on the ten coefficients $A, \ldots, K$, namely, the *determinant* **[153]**

$$\Delta = \begin{vmatrix} A & B & D & G \\ B & C & E & H \\ D & E & F & J \\ G & H & J & K \end{vmatrix} .$$

Since the coefficients $A, \ldots, K$ transform contragrediently to the quadratic terms in $\xi, \ldots, \tau$, it is easy to show that the *weight of this invariant* is -2: $\Delta' = r^{-2} \cdot \Delta$. *The full system of invariants formed alone from the coefficients of the form consists solely of this* Δ, i.e., *every* integral rational invariant, which contains only $A, \ldots, K$ is a multiple of a power of Δ.

If we add now the *coordinates* ξ, η, ζ, τ of a point to the coefficients of the former, the *simplest common invariant*, or (according to the terminology mentioned above) *covariant*, is the form f itself; for the transformations of the coefficients $A, \ldots, K$ are completely determined by the prescription of their invariance. Thus *every given form is of course its own covariant*. Indeed, by definition, it is entirely unchanged under our substitutions and is therefore an invariant of weight 0, or an *absolute invariant*. Moreover, if we employ two points $\xi_1, \ldots, \tau_1$ and $\xi_2, \ldots, \tau_2$, there will appear, as new covariant, the so-called *polar form*:

$$A\xi_1\xi_2 + B(\xi_1\eta_2 + \xi_2\eta_1) + C\eta_1\eta_2 + \cdots + K\tau_1\tau_2 ,$$

whose *weight is again 0*, i.e., it is likewise *absolutely invariant*.

Finally, if we consider, simultaneously with f, *also a linear form* ϕ, i.e., the totality of its coefficients $\alpha, \beta, \gamma, \delta$, we obtain the following *simultaneous invariant of weight* -2 which arises from the determinant through the so-called process of "bordering" with $\alpha, \beta, \gamma, \delta$:

$$\begin{vmatrix} A & B & D & G & \alpha \\ B & C & E & H & \beta \\ D & E & F & J & \gamma \\ G & H & J & K & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix} .$$

According to what precedes, we can also call it a *contravariant*. This determinant, as you know, plays an important role in analytic geometry. We recognise that the purely analytic process of forming invariants is fundamental here. **[154]**

If we have two linear forms ϕ_1, ϕ_2, with coefficients $\alpha_1, \ldots, \delta_1$ and $\alpha_2, \ldots, \delta_2$, we obtain, by a "double bordering" of the same determinant, another *invariant*:

$$\begin{vmatrix} A & B & D & G & \alpha_1 & \alpha_2 \\ B & C & E & H & \beta_1 & \beta_2 \\ D & E & F & J & \gamma_1 & \gamma_2 \\ G & H & J & K & \delta_1 & \delta_2 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & 0 & 0 \end{vmatrix},$$

which likewise has the *weight* -2.

These few statements must suffice to give you a glimpse of the broad field of the theory of invariants. An unusually extensive doctrine has been developed here, and much acumen has been exercised, especially in devising methods for setting up the complete system of invariants and the complete system of syzygies for a given fundamental form. Let me make just one more remark of a general character. In our examples, we have always reached our invariants by setting up determinants, and in this we find justification for the *theory of determinants as the foundation for the theory of invariants*. Because of this connection, Cayley originally used the name hyperdeterminants for invariants. It was Sylvester who introduced the word *invariant*. It is interesting to raise the question as to the importance, in the field of mathematics as a whole, which should be assigned to a particular chapter of it, let us say to determinants. Cayley once said to me, in conversation, that if he had to give fifteen lecture courses on the whole of mathematics, he would devote *one* of them to determinants. Reflect, if you will, whether, according to your experience, your appraisal of the value of the theory of determinants would be so high. I find that in my own elementary lecture courses, I have, for pedagogical reasons, pushed determinants more and more into the background. Too often I have had the experience that, while the students acquired facility with the schemata, which are so useful in abbreviating long expressions, they often failed to gain familiarity with their *meaning*, and habituation to the schema prevented the student from going into all the details of the subject and so gaining a mastery. Of course, in general considerations, and consequently here in the theory of invariants, determinants are indispensable.

We come now, at last, to our real object, to obtain, by the aid of these reflections, a systematisation of geometry.

[155] 3. Application of Invariant Theory to Geometry

Interpretation of Invariant Theory in Affine Geometry

We begin by interpreting the variables $\xi, \ldots, \tau$ as *ordinary rectangular non-homogeneous coordinates*: (ξ, τ) in the plane, (ξ, η, τ) in three-dimensional space,

(ξ, η, ζ, τ) in four-dimensional space, etc. The linear homogeneous substitutions of invariant theory

$$\begin{aligned} \xi' &= \alpha_1\xi + \cdots + d_1\tau\,, \\ &\cdots\cdots\cdots \\ \tau' &= \alpha_4\xi + \cdots + d_4\tau \end{aligned} \tag{1}$$

represent then *the⁻totality of affine transformations of the space under consideration with fixed origin of coordinates.* Each relative *invariant* itself will be a geometric quantity, which, to within a factor, remains unchanged by these affine transformations, i.e., *a quantity which has a definite meaning in the affine geometry defined by these transformations.*

If, for example, in the *binary case*, i.e., in the *plane*, two points 1 and 2 are given, then, as we have seen, the *fundamental invariant Δ_{12} represents twice the area of the triangle* (0 1 2), provided with a suitable sign. In fact, it is known (see the analogous situation for space, p. [78]) that an affine transformation merely multiplies the area of a triangle by the determinant of the substitution, and this means precisely that Δ_{12} is a relative invariant of weight 1. The quotient Δ_{12}/Δ_{34}, of two areas, remains absolutely unchanged, but so also does the *equation* $\Delta_{12} = 0$, since multiplication by a factor would have no significance in this equation. Actually, this equation has the absolutely invariant meaning, with respect to our affine transformation, that the three points $0, 1, 2$ lie on a straight line.

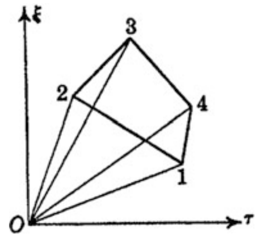

Figure 99

If we have several points $1, 2, 3, 4, \ldots$ (see Fig. 99), their *complete invariant system* consists of all their determinants Δ_{ik}. Hence if it is possible to construct a quantity which is a rational integer function of the coordinates and which is relatively invariant under all affine transformations (1), i.e., *which has significance*, *at all*, *in our affine geometry*, it must be expressible as a *polynomial in the Δ_{ik}*. We can verify this at once geometrically in simple cases, e.g., every area in the plane, say that of the polygon $(1, 2, 3, 4)$, is such an invariant, and the general formula which we gave earlier (p. [9]) for the area of a polygon

$$(1, 2, 3, 4) = \Delta_{12} + \Delta_{23} + \Delta_{34} + \Delta_{41}$$

is actually nothing but the expression of the general theorem for this special case.

[156] Finally let us consider the *syzygies* between the invariants. The fundamental syzygy

$$\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{42} + \Delta_{14}\Delta_{23} = 0$$

represents an identity between the areas of the six triangles formed by four arbitrary points and the origin, and therefore a general theorem of our affine geometry. Something similar holds, of course, for every syzygy. Conversely, every theorem of our affine geometry, insofar as it is a relation between invariants of the affine transformations (1), must be represented by a syzygy. Thus, according to our previous assertion (p. [152]) about the *complete* system of syzygies in the case of four points, *all* the theorems of our affine geometry, which are valid for a system of four points must follow from the one just given. In the same way, we can establish the correctness of the general assertion that *the theory of invariants permits the systematic enumeration of all possible quantities and theorems*, *without exception*, *since it supplies the complete system of invariants and syzygies*.

Again I shall refrain from carrying through this examination in detail. I mention merely that, along with points, one can consider also geometric configurations determined by forms $\phi = \alpha\xi + \delta\tau, f = A\xi^2 + 2G\xi\tau + \kappa\tau^2, \ldots$ Such a form sets up a correspondence between each point of the plane and a number, i.e., it determines a *scalar field*. With this point of view, we can easily interpret geometrically the invariants of a given form, and each syzygy between the invariants will represent again a geometric theorem.

Interpretation in Projective Geometry

Alongside of what I may call the naive interpretation of invariant theory in geometry of n dimensions, which we have thus far considered, in which the n variables are thought of as rectangular coordinates, there is another *essentially different interpretation*: One can think of the variables *as homogeneous coordinates* in $(n-1)$-dimensional space R_{n-1}, whose non-homogeneous coordinates are $x = \xi/\tau$, ..., where a factor common to the n coordinates is unessential. We discussed earlier (p. [87] et seq.) the connection between the coordinates in R_{n-1} and R_n. We thought of R_{n-1} as the linear $(n-1)$-dimensional configuration $\tau = 1$ of R_n and projected its points by rays drawn from the origin of R_n. The aggregate, then, of all possible systems of values of the homogeneous coordinates of a point in R_{n-1} is identical with that of the coordinates of the points in R_n corresponding to it. Now the linear substitutions of the homogeneous variables in R_{n-1} represent *projective*
[157] *transformations*. Indeed, all substitutions of the form

$$\begin{aligned}
&\rho'\xi' = a_1\xi + \cdots + d_1\tau\,,\\
&\cdots\cdots\cdots\\
&\rho'\tau' = a_4\xi + \cdots + d_4\tau\,,
\end{aligned}$$

which differ from one another by an arbitrary factor ρ' produce one and the same projective change. The *group of all these projective transformations* contains not n^2 but only $n^2 - 1$ *arbitrary constants*; in R_2 and R_3, in particular, the number of such constants is 8 and 15, respectively.

If we wish, then, to interpret the theory of invariants of n variables $\xi, \ldots, \tau$ geometrically in the projective geometry of R_{n-1}, we must bear in mind, above all, that, just because we are using homogeneous coordinates, only those quantities and relations of the theory of invariants will be capable of interpretation which are *homogeneous of order zero* in the coordinates $\xi, \ldots, \tau$ of every point that occurs, and which have the same property also with respect to every system of coefficients of a linear, quadratic, or other form which may occur.

This will become clear if I carry it out in concrete *examples*. It will be sufficient to discuss the *binary field* ($n = 2$). We assume, then, two variables ξ and τ, and we think of $x = \xi/\tau$ as an *abscissa on the straight line*. If a series of systems of values $(\xi_1, \tau_1), (\xi_2, \tau_2), \ldots$, is given, we know that the determinants

$$\Delta_{ik} = \begin{vmatrix} \xi_i & \tau_i \\ \xi_k & \tau_k \end{vmatrix} \qquad (i, k = 1, \ldots, p)$$

represent the complete system of fundamental invariants. Of all invariant statements, which ones have meaning in projective geometry? Among these is certainly not the statement that one of the Δ_{ik} has some definite numerical value, for if we multiply ξ_i, τ_i by a factor ρ, which would not change the point i, we multiply Δ_{ik} also by ρ. However, the vanishing of a Δ_{ik}, that is, the relation $\Delta_{ik} = 0$, has a meaning in projective geometry, for we can write it in the form $\xi_i/\tau_i = \xi_k/\tau_k$ so that actually only the ratios of the coordinates of the points appear, and the geometric significance – the *coincidence of the points i and k* – is evident.

In order, now, to get a *numerical invariant*, which is itself of dimension zero in the coordinates of each point, we must combine more than two points. Trial shows that we need at least four points 1, 2, 3, 4, in which case each quotient of the form

$$\frac{\Delta_{12} \cdot \Delta_{34}}{\Delta_{14} \cdot \Delta_{32}}$$

is homogeneous of dimension zero in each of the four pairs of variables $(\xi_1, \tau_1), \ldots,$ **[158]**
(ξ_4, τ_4). It follows from this that its weight is 0, i.e., it is an *absolute invariant*. This quantity has, then, a projective meaning and represents a numerical value, which is invariant under all projective transformations of the line. It is, of course, nothing other than the *cross-ratio* of the four points written in a definite order. For it can be written, in non-homogeneous coordinates, in the form

$$\frac{x_1 - x_2}{x_1 - x_4} : \frac{x_3 - x_2}{x_3 - x_4}.$$

From the standpoint of the theory of invariants, we obtain the cross-ratio of four points as the simplest invariant of a point series on the straight line, which sat-

isfies the homogeneity condition that is necessary in order that the invariant have a meaning in projective geometry.

I should like to add here a general remark. Earlier on, I have thought about the widespread tendency in projective geometry to resolve all quantities, which exhibit invariant character back to cross-ratios. From the standpoint, which we have reached, we can pronounce the judgment that such an effort only makes it more difficult to gain a deeper insight into the structure of projective geometry. It is better to begin with a search for all rational integer (relative) invariants and to form from them, first, the rational invariants, especially the absolute ones, and among these again those, which satisfy the homogeneity condition of projective geometry. In this way we follow a systematic procedure, which progresses from the simplest to the more complex. This procedure is obscured if we place in the foreground a special rational invariant, the cross-ratio, and try to form the other invariants exclusively from it.

Let us now see to what kind of theorems of projective geometry the *syzygies* between the invariants $\varDelta_{ik}$ give rise. Starting from the fundamental syzygy

$$\varDelta_{12}\varDelta_{34} + \varDelta_{13}\varDelta_{42} + \varDelta_{14}\varDelta_{23} = 0\,,$$

dividing through by the last summand of the left side, and noting that $\varDelta_{23} = -\varDelta_{32}$, and $\varDelta_{24} = -\varDelta_{42}$, we get

$$\frac{\varDelta_{12}\varDelta_{34}}{\varDelta_{14}\varDelta_{32}} = 1 - \frac{\varDelta_{13}\varDelta_{24}}{\varDelta_{14}\varDelta_{23}}\,.$$

Here we have, on the left, the cross-ratio of the points 1, 2, 3, 4, according to the original definition. On the right, we have the cross-ratio of the same points formed in the same way after the order of 2 and 3 has been changed. The cross-ratios in still other orders are obtained if we divide by other terms. *Thus the fundamental*
[159] *syzygies between the invariants of four points find their geometric meaning in the known relations between the six values, which their cross-ratio can take according to the order in which the four points are taken.*

I shall not go any farther here in showing how the projective geometry of the straight line is built up on this foundation and how, in like manner, the *interpretation of the ternary and quaternary theory of invariants in the projective geometry of the plane and of space* proceeds. You will find that set forth in detail in, for example, the books of Salmon-Fiedler and Clebsch-Lindemann, already mentioned, where precisely this interpretation of the theory of invariants is used continually. There arises thus a *self-contained complete development of projective geometry*, not only with respect to the quantities, which one can consider in it (corresponding to the invariants), but also with respect to the theorems which can be set up (corresponding to the syzygies). To be sure, this interpretation is less satisfying for the student of invariants than it is for the geometer. For the former, the interpretation given in the study of affine geometry of R_{n+1} is more valuable, since in R_n only those invariants and syzygies are useful which satisfy the homogeneity condition, as we have seen.

I should like to consider in more detail one especially important point, in order to resume the discussion, which we interrupted earlier (p. [146]). I should like to show how the Cayley principle makes it possible by use of the theory of invariants to classify affine and metric geometry in the scheme of projective geometry.

4. The Systematization of Affine and Metric Geometry Based on Cayley's Principle

We are concerned here with *general* affine geometry, where we do not assume a special fixed point, the origin of coordinates, as was the case when the complete interpretation of the theory of invariants was first discussed.

We start at once, in three-dimensional space, with the non-homogeneous coordinates x, y, z or, as the case may be, with the homogeneous coordinates ξ, η, ζ, τ. Then the Cayley principle states that *affine* geometry or *metric* geometry arises from projective geometry when we adjoin to the given configuration the *plane at infinity*, $\tau = 0$, or this plane and also the imaginary *spherical circle* $\tau = 0$, $\xi^2 + \eta^2 + \zeta^2 = 0$, respectively.

A *remark about the imaginary spherical circle* will simplify the following discussion. We have defined it here by two equations, as the intersection of the plane at infinity with a cone through the origin. But we can determine it, or, in fact, any conic section, also by *one equation in plane coordinates*, if we think of it as the *envelope* of all the planes, which touch it. If, as before, we denote the "plane co- **[160]**
ordinates," i.e., the coefficients of a linear form ϕ, by α, β, γ, δ, then, as is easily verified, the equation of the imaginary spherical circle is $\alpha^2 + \beta^2 + \gamma^2 = 0$. In other words, this equation is the condition that the plane $\alpha\xi + \cdots + \delta\tau = 0$ shall be tangent to the imaginary spherical circle.

Subsumption of the Basic Concepts of Affine Geometry Under the Projective System

It is now easy to understand the transition by means of the theory of invariants to affine and to metric geometry, respectively. To the given systems of values – point coordinates, linear and quadratic forms, etc. – which describe the configuration under discussion, we add the *definite linear form r* (i.e., the system of coefficients $0, 0, 0, 1$), or *the quadratic form* $\alpha^2 + \beta^2 + \gamma^2$, *written in plane coordinates*, respectively. If, just as before, we treat the system of forms thus extended, i.e., if we set up *the full system of its invariants and of the syzygies between these*, and emphasise those among them which satisfy the *condition of homogeneity*, we obtain *all of the concepts and all of the theorems of affine and of metric geometry*, *respectively*, *of the elements originally given*. The development by means of the theory of invariants is thus carried over to affine and to metric geometry. I should like again to call your

attention (see p. [158]) to the fact that, by emphasising in particular the forming of *rational integer* invariants and syzygies, a systematising point of view comes into geometry, which otherwise is not much emphasised.

Instead of talking abstractly about this, I prefer to make these relations clear at once by means of *simple examples* by showing how we can represent the most elementary fundamental quantities of affine and metric geometry as simultaneous invariants of the given systems of quantities and of the forms τ and $\alpha^2 + \beta^2 + \gamma^2$, respectively.

To start with, I choose from *affine geometry*, as an example, the *volume T of the tetrahedron formed by four points*, which, as you know, is expressed by the formula

$$T = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = \frac{1}{6\tau_1\tau_2\tau_3\tau_4}\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 \\ \xi_3 & \eta_3 & \zeta_3 & \tau_3 \\ \xi_4 & \eta_4 & \zeta_4 & \tau_4 \end{vmatrix}.$$

Let us inquire to what extent this expression has the asserted invariant property. In the first place, we know that this determinant is actually the fundamental relative invariant of four points (p. [152]). Moreover, we find, in the denominator for these four points, the values of the linear form τ, which we adjoined to our configuration, and these are the very simplest (absolute) invariants that can be constructed by the
[161] use of a form (p. [153]). This means, of course, that, after a transformation, those values of the form into which the linear form τ goes over are to be written in the denominator, or that, if we adjoin in general the form $\alpha\xi + \beta\eta + \gamma\zeta + \delta\tau$, the product of the four values of this form for the points $1, \ldots, 4$ is to go into the denominator. *Thus T is itself also a rational invariant* and, indeed, it is homogeneous of dimension zero in the coordinates of each of the four points. To be sure, *T* has the dimension -4 with respect to the coefficients of our adjoined linear form 0, 0, 0, 1 (or α, β, γ, δ, as the case may be), which appear in the denominator. Hence, since a common factor of these quantities is arbitrary, the absolute value of *T* can have no meaning in the projective geometry of our extended figure. In fact, there is also no way of assigning a definite numerical value to the volume of a tetrahedron in affine geometry, unless we have already selected a unit segment or a unit tetrahedron, as we always did when we were using non-homogeneous coordinates. But this would mean, from our present general point of view, that we should add to our figure other elements beside the "infinitely distant plane" $\tau = 0$. If we adjoin a fifth point, for example, and take the *quotient* of two expressions analogous to *T*, we have actually an expression that satisfies all of the conditions of homogeneity. This expression must be, then, an *absolute invariant* of affine geometry. The single expression *T* is only a *relative* invariant of *weight* 1, as indeed we learned earlier (see p. [78]).

Subsumption of the Graßmannian Determinant Principle Under Invariant Theory; Tensors

At this point we should refer again to the *developments of the first main part*, the essential meaning of which now appears more clearly. We recognised in our special study of affine transformations (see pp. [77]–[78]) that the Graßmann elementary quantities of geometry, which we deduced there belong entirely to affine geometry. The *Graßmann determinant principle*, however, which supplied those quantities, is by no means a haphazard device. To the contrary, as we can now see, it is a thoroughly *natural application of the theory of invariants in affine geometry*, i.e., projective geometry under adjunction of the plane at infinity. The appearance of the ordinary determinants – segment, area, volume – is sufficiently explained by the example just discussed. It remains to be shown how the development by the theory of invariants leads to the general Graßmann elements defined by the minors of *rectangular matrices*. That, again, will be made clear by means of an example. Given two points (ξ_1, η_1, τ_1) and (ξ_2, η_2, τ_2) in a plane, we wish to find the equivalent in the theory of invariants of the configurations of affine geometry (line segment, straight line, ...), which belong to them. This falls into orderly agreement with earlier results if we add a third "undetermined" point $\xi, \eta, \tau)$ and consider **[162]**
again the fundamental invariant

$$\frac{1}{\tau\tau_1\tau_2}\begin{vmatrix} \xi & \eta & \tau \\ \xi_1 & \eta_1 & \tau_1 \\ \xi_2 & \eta_2 & \tau_2 \end{vmatrix}$$

as a linear form in ξ, η, τ. The three coefficients of these variables, that is, the determinants of the matrix

$$\frac{1}{\tau_1\tau_2}\begin{vmatrix} \xi_1 & \eta_1 & \tau_1 \\ \xi_2 & \eta_2 & \tau_2 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix},$$

are thus the characteristic magnitudes for the newly defined manifold, and *we have actually been led precisely to the matrix, which was used earlier to define the line segment* 1 2. In exactly the same way, in space, we can set up, from three or from two points, by adjoining one or two quadruples of undetermined coordinates, respectively, a relatively invariant linear or bilinear form, whose coefficients then supply the coordinates of a plane segment or a space line segment, in entire agreement with our old definition. I cannot amplify these suggestions with further details; they will perhaps suffice as a first orientation and as stimulation to further study.

Now that we have found an ordered place in the theory of invariants for the principle of Graßmann, it is more important to raise the question as to its achievement potential. In this connection, we should compare it especially with that principle of classification which was stated (pp. [27]–[28]) for the particular case of the principal group, and which yielded for us there all the fundamental geometric configurations. The appropriate extension of the principle of classification to the case of an arbitrary linear transformation group is obvious. According to it, we shall consider, in

each "geometry," alongside of the *individual* rational integer functions of the given series of quantities (coordinates, form coefficients, etc.), which thus far have furnished the invariants, also *systems* of such functions $\Xi_1, \Xi_2, \ldots$. If such a system is transformed into itself under all the substitutions of the pertinent group concerned, i.e., if the similarly formed functions $\Xi_1', \Xi_2', \ldots$ of the transformed series of quantities are expressed linearly in terms of $\Xi_1, \Xi_2, \ldots$ alone, with the aid of coefficients, which arise in a definite and unique manner from those of the fundamental transformation, we say *that the system defines a configuration of the geometry in question*. The separate functions of which the system consists are called the *components* of the configuration. The decisive property, which determines the nature of a geometric configuration is the behaviour of its components under the transformations of the group under consideration. Two geometric configurations are said to be of
[163] the *same sort* when their components form two series of the same number of expressions, each of which, under change of coordinates, undergoes the *same* linear substitution, that is, they are *cogredient*, according to our earlier terminology. If the system, which defines a geometric configuration consists of a single function, the linear substitution reduces to a multiplication by a factor, and the function is a relative invariant.

I shall make this abstract situation clear by means of a simple example from the invariant theory of the ternary field, which we shall interpret in the affine geometry of three-dimensional space with a fixed origin. If two points (ξ_1, η_1, τ_1) and (ξ_2, η_2, τ_2) are given, then the simplest system of functions in which both coordinate triples appear homogeneously and symmetrically is the system of nine bilinear terms

$$\xi_1\xi_2, \xi_1\eta_2, \xi_1\tau_2, \eta_1\xi_2, \ldots, \tau_1\tau_2\,. \tag{1}$$

Under a linear transformation, in our customary notation (see p. [147]), we get:

$$\left\{\begin{array}{l} \xi_1'\xi_2' = a_1^2\xi_1\xi_2 + a_1b_1\,(\xi_1\eta_2 + \eta_1\xi_2) + \cdots + d_1^2\tau_1\tau_2\,, \\ \xi_1'\eta_2' = a_1a_2\xi_1\xi_2 + a_1b_2\xi_1\eta_2 + a_2b_1\eta_1\xi_2 + \cdots + d_1d_2\tau_1\tau_2\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_1'\tau_2' = a_4^2\xi_1\xi_2 + a_4b_4\,(\xi_1\eta_2 + \eta_1\xi_2) + \cdots + d_4^2\tau_1\tau_2\,, \end{array}\right. \tag{2}$$

i.e., these nine quantities form, in fact, a system of the sort just discussed. We shall look upon them as the determining elements of a configuration of our affine geometry. Such a configuration, and likewise any other system consisting of nine quantities which transform according to the equations (2), is called a *tensor*.

Upon examining equations (2), we notice that we can derive from the nine quantities (1), on the one hand six, and on the other hand three, simple linear combinations, which are transformed into themselves under a linear substitution. Indeed, if we arrange the quantities (1) into a quadratic system

$$\begin{array}{lll} \xi_1\xi_2 & \xi_1\eta_2 & \xi_1\tau_2\,, \\ \eta_1\xi_2 & \eta_1\eta_2 & \eta_1\tau_2\,, \\ \tau_1\xi_2 & \tau_1\eta_2 & \tau_1\tau_2\,, \end{array}$$

the first set is the sums of the terms symmetric to the diagonal:

$$(3) \qquad 2\xi_1\xi_2, \xi_1\eta_2 + \eta_1\xi_2, \xi_1\tau_2 + \tau_1\xi_2, \ldots, 2\tau_1\tau_2\,,$$

and the other is their differences:

$$(4) \qquad \xi_1\eta_2 - \eta_1\xi_2,\quad \xi_1\tau_2 - \tau_1\xi_2,\quad \eta_1\tau_2 - \tau_1\eta_2\,.$$

The substitution formulas for the systems (3) and (4) come immediately from equations (2). Thus we have secured two new configurations for our affine geometry, **[164]** of which the one, made up of the six quantities (3), is called a *symmetric tensor*, while that consisting of the three quantities (4) is the *plane segment* already known to us. The name applies, of course, to any system of quantities, which are transformed cogrediently. We shall consider in short the justification for the adjective "symmetric."

As to the geometric meaning of the three quantities (4), we know (see p. [32]) that they are twice the projections upon the coordinate planes of the triangles formed by the points (ξ_1, η_1, τ_1) and (ξ_2, η_2, τ_2), and the origin of coordinates, each triangle contour being traversed in a suitable sense. We have here precisely one of the first configurations, which the Graßmann determinant principle yielded. Hence we may enunciate the following theorem. *The systematic search for configurations of affine geometry by means of our principle of classification leads necessarily*, *among other things*, *to the Graßmann determinant principle and to the geometric configurations determined by its use*. Of course, I cannot carry this out here in detail. It will suffice to state that all the configurations can be derived which we discussed earlier if we treat the general affine geometry in a similar way by means of Cayley's principle, by means of the quaternary invariant theory (see pp. [160] sqq.).

The important result of our examination, however, is the knowledge that *the Graßmann determinant principle is something special*, *and*, *in itself*, *does not at all yield all the configurations of affine geometry*. We have, rather, in the tensors (1) and (3) *essentially new geometric quantities*.

Because of the great significance, which these configurations have for many fields of physics, as, for example, for the theory of elastic deformation and for the theory of relativity, I shall discuss them briefly. Above all, I shall make some remarks concerning the names of these quantities, which should help the reader to orient himself in the newer literature on tensor calculus. I used the word *tensor* in volume 1 of this work, when I was discussing Hamilton's quaternion calculus, in a sense different from that which we are now using. If $q = a + bi + cj + dk$ is a quaternion, we called the expression $T = \sqrt{a^2 + b^2 + c^2 + d^2}$ its tensor. This name, introduced by Hamilton, is justified, since one can interpret multiplication by a quaternion, geometrically, as a rotation and a stretching, with a fixed origin, as we explained fully in volume 1 (pp. [71] sqq.). The measure of the stretching turns out to be precisely the square root T, which we called the tensor. Woldemar Voigt,

[165] in his work on the physics of crystals,[63] used the word *tensor* in a manner closely related to this. Voigt denotes by it directed quantities, which correspond to events, such as the longitudinal stretching or compression of a rod, at the ends of which pulls or pushes are applied in the direction of the axis of the rod, but in opposite senses. We could represent such a tensor pictorially by a segment, which carries at its ends arrowheads oppositely directed (see Fig. 100).

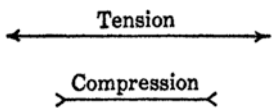

Figure 100

We could designate the directional character of a tensor, thus understood, as "two-sided," and that of a vector, by contrast, as "one-sided." Such tensors arise often in physics as *tensor triples*, i.e., three of them at right angles to one another (see Fig. 101). We mentioned earlier (see p. [80]) a pure strain (pure affine transformation) as a uniform stretching of space in three mutually orthogonal directions, with a fixed origin. Instead of this, we can say now that a pure strain is represented geometrically by a tensor triple. We reach a commonly used meaning of the word *tensor* if we think of the concept of those three stretchings of space as a single geometric quantity, and, dropping the word *triple*, call *this* quantity a tensor. The tensor notion in this sense is precisely what we called above a "symmetric tensor."

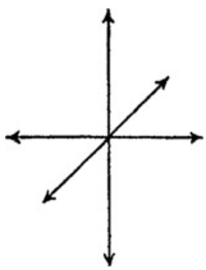

Figure 101

In fact, a pure strain, with a fixed origin, is given by substitutions of the following form

$$(5) \qquad \left\{ \begin{aligned} \xi &= a_{11}x + a_{12}y + a_{13}z\,, \\ \eta &= a_{12}x + a_{22}y + a_{23}z\,, \\ \tau &= a_{13}x + a_{23}y + a_{33}z\,. \end{aligned} \right. \qquad (a_{ik} = a_{ki})\,.$$

[63] See, for example, (a) *Der gegenwärtige Stand unserer Kenntnisse der Kristallelastizität*; (b) *Über die Parameter der Kristallphysik und über gerichtete Grössen höherer Ordnung*. Both memoirs in the *Göttinger Nachrichten* 1900.

Let us interpret the number triples (x, y, z) and (ξ, η, τ) as point coordinates in one and the same rectangular coordinate system. The array of the coefficients of the transformation is symmetrical with respect to the principal diagonal. If we go over now to a new rectangular coordinate system with the same origin, we obtain, as a simple calculation shows, the following new representation for the strain in question: **[166]**

$$(6)\qquad \begin{cases} \xi' = a'_{11}x' + a'_{12}y' + a'_{13}z' , \\ \eta' = a'_{12}x' + a'_{22}y' + a'_{23}z' , \\ \tau' = a'_{13}x' + a'_{23}y' + a'_{33}z' . \end{cases} \qquad \left(a'_{ik} = a'_{ki}\right) .$$

The same formulas give the relations between x, y, z and x', y', z' as between ξ, η, τ and ξ', η', τ'. For the six coefficients $a'_{11}, a'_{12}, \ldots, a'_{33}$ it turns out that

1. They depend linearly upon the six coefficients a_{11}, a_{12}, …, a_{33}, and upon these only, i.e., *they define a geometric quantity*.

2. They transform precisely as do the expressions (3), bilinear in the coordinates, which we designated on p. [164] as the components of a *symmetric tensor*. The adjective *symmetric* is justified by the form of the array of coefficients in the transformation formulas (5) and (6).

Let us now go over to the general affine transformation

$$(7)\qquad \begin{cases} \xi = a_{11}x + a_{12}y + a_{13}z , \\ \eta = a_{21}x + a_{22}y + a_{23}z , \\ \tau = a_{31}x + a_{32}y + a_{33}z , \end{cases}$$

where the origin is left fixed. Then it yields, in a manner corresponding precisely to that just indicated, that in the geometry of the orthogonal transformations the nine coefficients a_{11}, a_{12}, …, a_{33} transform precisely as do the coordinate products (1); hence they form the components of a quantity of the same sort. This means, in our terminology, according to which the word *tensor* is not restricted especially to pure strains, that the array of coefficients of a general affine transformation is a tensor.

A large number of other names for this concept are to be found in the literature. Some of the most common are the following.

1. *Affinor* (because of the connection with the affine transformation).

2. *Linear vector function* [since the linear substitutions (7) can be so interpreted that, by means of them, to a vector x, y, z, starting from the origin, another similar vector ξ, η, τ will be placed in linear correspondence].

3. *Dyad* and *dyadic*. However, the first of these two words is used originally only for a particular case, to be explained later.

The components of the plane segment (4) also can be regarded as the coefficients of a transformation, namely one of the type

$$(8)\qquad \begin{cases} \xi = 1 \cdot x - c \cdot y + b \cdot z , \\ \eta = c \cdot x + 1 \cdot y - a \cdot z , \\ \tau = -b \cdot x + a \cdot y + 1 \cdot z . \end{cases}$$

[167] Indeed, it is easy to show that the coefficients of this substitution behave, under rectangular coordinate transformation, as do the bilinear expressions (4). Because of the structure of the array of coefficients in (8) (symmetry with respect to the main diagonal along with change of sign), the quantity determined by it is also called an *antisymmetric tensor*.

Geometrically, the formulas (7) can be interpreted as a general homogeneous deformation, the formulas (6) as a pure deformation (without rotation), and the formulas (8) as an *infinitesimal* rotation. The *decomposition of a homogeneous infinitesimal deformation into a pure deformation and a rotation corresponds thus perceptually to the formal process* (*p*. [151]) *in which we derived the symmetric tensor* (3) *and the antisymmetric tensor* (4) *from the coordinate products* (1).

Thus far, in changing the coordinate system, we have confined ourselves to orthogonal transformations. It remains to complete this for the case in which we pass from the rectangular to oblique coordinates, or, indeed, where both (ξ, η, τ) and (x, y, z) are, at the start, introduced as oblique parallel coordinates. We shall continue to think of the origin of coordinates as fixed. In making this change, we pass from the geometry of the principal group to that of the affine group. When we examine, for this group, the behaviour of the substitution coefficients under transformation of the coordinates, it turns out that, although they again represent the components of a geometric quantity, they are transformed, not as are the coordinate products (1), but contragrediently to them. The coefficients of (6) and (8) behave in a corresponding way. It can be shown that the same tensor (for example the same homogeneous deformation) with respect to a parallel coordinate system can be given by components of the kind (1), as also by such components as the coefficients of (7). The former are called *cogredient*, the latter are called *contragredient* components of the tensor. Instead of *cogredient* and *contragredient*, the terms *contravariant* and *covariant* are often used. Sometimes the last two expressions are interchanged in meaning. The difference between the two kinds of components is the same as that between point and plane coordinates.

Another meaning of the word *tensor*, and one that is much more general than the one we have favoured, will become clear if we study the behaviour of homogeneous forms under a change of coordinates. On p. [138], we carried through this investigation for the case of a quadratic form

$$a_{11}\xi^2 + 2a_{12}\xi\eta + \cdots + a_{33}\tau^2\,,$$

using a somewhat different notation. We found that the form coefficients a_{11}, $2a_{12}$,
[168] ..., a_{33} transform linearly, homogeneously, and contragrediently to the terms ξ^2, $\xi\eta$, ..., τ^2 of the point coordinates. The latter, however, transform cogrediently to the expressions (3), as it is easy to see. We can announce this result as follows. The coefficients $a_{11}, 2a_{12}, \ldots, a_{33}$ of a quadratic form are the contragredient components and the terms ξ^2, $\xi\eta$, ..., τ^2 are the cogredient components, of a symmetric tensor. A corresponding result holds for a bilinear form. Of the latter, we may say, with Gibbs, that it forms a *dyad* when it can be written as a product of two linear forms. Finally if we have *a homogeneous n-tuple linear form* of the point coordinates, we

can show, by a slight calculation, that its coefficients likewise substitute linearly and homogeneously under transformation of coordinates, and, indeed, contragrediently to the terms of the point coordinates.

The generalisation of the tensor notion, which we have discussed, consists in calling every such quantity a tensor, using this name not merely, as we did before, in connection with bilinear forms. It is in this general form that the name is used, in particular, by Albert Einstein and his disciples. In the older terminology it was customary to speak rather of linear, quadratic, bilinear, trilinear, cubic, etc., *forms*.

Along with this variety in terminology, there appears the tendency, in practice, to denote the system of components of a tensor by a *single* letter, and to indicate calculations with tensors, when they arise, by means of symbolic combinations of the letters. All these things are in themselves essentially very simple; if they seem difficult to the reader, it is only because different writers use different notations. The same unfortunate situation arises here that we mentioned when we were discussing the vector calculus, but here it is greatly exaggerated. However, it seems impossible to get rid of the confusion. We could not refrain from mentioning it, since the whole modern literature is dominated by it.

Subsumption of Metric Geometry Under the Projective System

Let us now turn to *metric geometry* in order to select there a few characteristic examples. I shall show how the two most important fundamental notions "*distance r between two points* $x_1 = \xi_1/\tau_1, \ldots$, and $x_2 = \xi_2/\tau_2, \ldots$," as well as "*angle ω between two planes* $\alpha_1, \ldots, \delta_1$ and $\alpha_2, \ldots, \delta_2$" can be derived from the systematic procedure of the theory of invariants. From the well-known formulas of analytic geometry, we have

$$\begin{aligned} r &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\ &= \sqrt{\frac{(\xi_1\tau_2 - \xi_2\tau_1)^2 + (\eta_1\tau_2 - \eta_2\tau_1)^2 + (\zeta_1\tau_2 - \zeta_2\tau_1)^2}{\tau_1^2\tau_2^2}}, \\ \omega &= \arccos\left(\frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2}{\sqrt{\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right)\left(\alpha_2^2 + \beta_2^2 + \gamma_2^2\right)}}\right). \end{aligned}$$

These are algebraic and transcendental functions, respectively, of the parameter. **[169]** We may call them "*algebraic*" and "*transcendental*" *invariants*, respectively, if we show that the rational integer parts of which they are formed are themselves invariants in the old sense.

We start with the angle ω. The figure, whose invariant it should be, consists of the two linear forms $\alpha_1, \beta_1, \gamma_1, \delta_1$ and $\alpha_2, \beta_2, \gamma_2, \delta_2$, and the quadratic form in plane

coordinates

$$\alpha^2 + \beta^2 + \gamma^2 + 0 \cdot \delta^2 \,,$$

which represents the imaginary spherical circle. We can of course construct invariants from this quadratic form in plane coordinates, just as we did earlier (pp. [152] sqq.) from forms in point coordinates, by always interchanging point and plane coordinates ("dualising"). In particular, the values of the form for the two given systems of values

$$\alpha_1^2 + \beta_1^2 + \gamma_1^2 + 0 \cdot \delta_1^2 \quad \text{and} \quad \alpha_2^2 + \beta_2^2 + \gamma_2^2 + 0 \cdot \delta_2^2$$

and also the value of the polar form constructed for these two systems

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + 0 \cdot \delta_2\delta_2$$

are all invariant. It is precisely out of these expressions that cos ω is actually constructed. Furthermore, $\cos\omega$ is *homogeneous of dimension zero* in each of the two systems $\alpha_1, \ldots, \delta_1$ and $\alpha_2, \ldots, \delta_2$, and likewise in the coefficients $1, 1, 1, 0$ of the given quadratic form, so that the expression has an independent meaning in metric geometry. There is, in fact, in metric geometry, an absolute angle measure, which is independent of the arbitrary choice of the unit. This amounts to saying that our expression is an *absolute invariant*.

Next, as to the *distance* r, we recall that we constructed invariants of a quadratic form in point coordinates by bordering its determinant with the coordinates of one or of two planes (see pp. [153]–[154] sqq.). In the same way we shall now obtain invariants for our figure, which consists of a quadratic form in plane coordinates and two points, if, proceeding precisely in a dual manner, we border the determinant of the form $\alpha^2 + \beta^2 + \gamma^2 + 0 \cdot \delta^2$:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

[170] once and twice with the coordinates $\xi_1, \ldots, \tau_1$ and $\xi_2, \ldots, \tau_2$ of the given points. From the determinants thus obtained we form the quotient

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \xi_1 & \xi_2 \\ 0 & 1 & 0 & 0 & \eta_1 & \eta_2 \\ 0 & 0 & 1 & 0 & \zeta_1 & \zeta_2 \\ 0 & 0 & 0 & 0 & \tau_1 & \tau_2 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 & 0 & 0 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 & 0 & 0 \end{vmatrix} : \left\{ \begin{vmatrix} 1 & 0 & 0 & 0 & \xi_1 \\ 0 & 1 & 0 & 0 & \eta_1 \\ 0 & 0 & 1 & 0 & \zeta_1 \\ 0 & 0 & 0 & 0 & \tau_1 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 & \xi_2 \\ 0 & 1 & 0 & 0 & \eta_2 \\ 0 & 0 & 1 & 0 & \zeta_2 \\ 0 & 0 & 0 & 0 & \tau_2 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 & 0 \end{vmatrix} \right\}.$$

If we develop these three determinants, it is easy to show that this quotient is precisely the value given above for r, which is thus shown to be invariant. Like the

fundamental invariant of affine geometry, which we considered earlier, this quotient is homogeneous and of dimension zero in the coordinates of the two given points, but not in the coefficients of the given quadratic form, in which it is homogeneous and of dimension -4. Moreover it is *not an absolute invariant*, for each of the determinants has the weight 2, i.e., the quotient has the weight $2 - 4 = -2$, as we see from the fact that what we here have is the dual of the constructions considered on pp. [153]–[154]. Consequently the numerical value of r has no immediate significance in metric geometry. Indeed, we can measure the distance between two points only if we assume a further arbitrary (unit) segment, i.e., if we adjoin that segment to the figure, along with the fundamental quadratic form. Absolute invariants of metric geometry appear only if we construct quotients of expressions of the sort here considered.

Here again I must not go into further detail. These examples will give you, at least, some idea as to the appearance of the complete systematic development of affine and metric geometry, which results from the systematic articulation of rational integer invariants. I hope that you will extend your knowledge by reading in the many textbooks already mentioned.[64]

Projective Treatment of Triangle Geometry

I shall touch a certain simple example, which is treated in detail in the new edition of Clebsch-Lindemann;[65] I refer to the so-called *geometry of the triangle*. In the course of time, an extensive closed field has emerged here, due especially to the work of Gymnasium teachers, devoted to the many *remarkable points*, *straight lines*, *circles*, which can be defined in connection with the triangle: the centre of **[171]** gravity, the altitudes, the bisectors of the angles, the incircles, the circumcircle, the Feuerbach circle, and so on. The countless relations, toward the discovery of which men have long striven, and are still striving, fall easily into orderly arrangement in our systematic structure. Let there be *given*, as vertices of a triangle, *three points*

$$(\xi_1, \eta_1, \tau_1)\,, \quad (\xi_2, \eta_2, \tau_2)\,, \quad (\xi_3, \eta_3, \tau_3)\,.$$

Since we are concerned with metric relations, we adjoin the two *imaginary circular points*, whose line equation is $\alpha^2 + \beta^2 = 0$. We may simply adjoin the values $(1, i, 0)$ and $(1, -i, 0)$ of their point coordinates. (See Fig. 102.) Then the whole *geometry of the triangle is nothing else than the projective invariant theory of these 5 points*, i.e., five arbitrary points, two of which we denote by special terminol-

[64] [In connection with the above, attention should be called especially to a paper by Heinrich Burkhardt in vol. 43 (1893) of the *Mathematische Annalen*: *Über Funktionen von Vektorgrössen, welche selbst wieder Vektorgrössen sind. Eine Anwendung invariantentheoretischer Methoden auf eine Frage der mathematischen Physik*.]

[65] *Loc. cit.*, p. 321. I should mention, above all, the *Enzyklopädie* report by Berkhan & Meyer on newer triangle geometry (III A B 10).

ogy. This remark only gives to geometry of the triangle the transparent character of a systematic teaching structure, which is otherwise lost to sight.

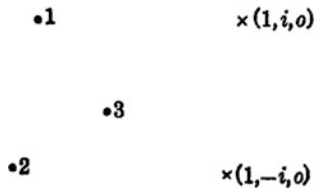

Figure 102

With this I leave the consideration of the systematic structure of geometry. It certainly satisfies the aesthetic sense to have an orderly arrangement of the sort, which I have described. Moreover, since this systematisation alone permits a deeper insight into geometry, every mathematician, every prospective teacher, should know something about it. For this reason I felt compelled to include it in this course, although you will often find this point of view in the literature, but perhaps not always in such a consistent presentation. Of course it would be entirely perverse to tie us dogmatically to this systematisation and to present geometry always in this light. The subject would soon become tedious and would lose all attractiveness. Above all, this would be a bar to investigative thought, which always functions independently of systematic planning.

Up to this point we have been considering, in a sense, the architecture of the structure of geometry. We shall now turn our attention to the no less important question of its foundations.

II. Foundations of Geometry

An overview of the very extensive field which we now enter is afforded by the *Enzyklopädie* report by *Federigo Enriques* entitled *Prinzipien der Geometrie* (Enz. III A. B. 1). Investigations in the foundations of geometry often approach very closely the interests of the theory of knowledge and of psychology, which, from their viewpoints, study the origin of space intuition and the justification of treating it by mathematical methods. We shall touch these questions very superficially, **[172]** of course, and we shall treat essentially the *mathematical side of the problem*, assuming that space intuition is to be taken for granted. We must also pass over the question that is so important in pedagogy, as to how space intuition develops in the individual to the precise form to which we, as mathematicians, are accustomed.

General Problematic; Relation to Analytic Geometry

Our problem, restricted in this manner, is to *erect the entire structure of geometry upon the simplest foundation possible*, *by means of logical operations*. Pure logic cannot, of course, supply the foundation. Logical deduction can be used only after the *first part* of the problem is solved, i.e., *after we have a system which consists of certain simple fundamental notions and certain simple statements* (*the so-called axioms*), *and which is in accord with the simplest facts of our intuition*. These axioms may be subdivided, of course, according to taste, into separate components, which are independent of one another. Otherwise we have great freedom in choosing them. The one condition which the system of axioms must satisfy is imposed by the *second part* of our problem: *It must be possible to deduce the entire contents of geometry logically from these fundamental notions and axioms*, without making any further appeal to intuition.

The conception of this lecture course suggests a definite characteristic way of treating this question. As a matter of principle, we have always availed ourselves of the aids of analysis, and in particular of the methods of analytic geometry. Hence we shall here again assume knowledge of analysis, and we shall inquire *how we can go*, *in the shortest way*, *from a given system of axioms to the theorems of analytic geometry*. This simple formulation is, unfortunately, rarely employed, because

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_13

geometers often have a certain aversion to the use of analysis, and desire, insofar as possible, to get along without the use of numbers.

Hints Regarding the Construction of Projective Geometry, with Subsequent Connection of Metric Geometry

The programme thus indicated in general can be carried through in different ways, depending upon *which* fundamental notions and axioms we decide to use. It is convenient, and not unusual, to start with the *fundamental notions of projective geometry*, namely, with *point*, *straight line*, *and plane*, which we have already emphasised as fundamental concepts (pp. [62]–[63]). We should not try to set up *definitions* as to what sort of things these are – one must know that from the start! The programme demands rather a statement of only so many characteristic properties and mutual relations that we can derive from them, in the sense indicated above, the whole of geometry. I shall not enumerate completely the separate axioms that would suffice for this purpose, for that would carry us too far afield. I shall only
[173] characterise their contents sufficiently for you to get a clear idea of them.

At the head are the *theorems of connection*, which I enunciated earlier (p. [63]) for projective geometry. We shall not demand, at the outset, as we did there, the existence, without exception, of a point of intersection of two straight lines in a plane or of a straight line of intersection of two planes. Instead, as befits the relations of metric and affine geometry, we shall restrict ourselves to the theorem that *two straight lines of a plane have one point*, *or none*, *in common*, *two planes have either a straight line or else not a single point in common*. We can then derive, by the adjunction of "improper" points, straight lines, and planes, the complete system of projective geometry.

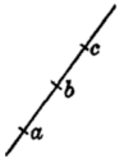

Figure 103

Next come the *theorems of order*, which describe how different points in the plane and on the straight line can lie with respect to each other. Thus, of three points a, b, c on a straight line, there is always one, say b, which lies between the other two, a and c; and so on. These statements are also called *theorems of betweenness*. (See Fig. 103.)

Finally, as to properties of continuity, I shall emphasise here, for the present, only the fact that the straight line has no gaps in it. If we separate, in any way, the segment between two points a and b into two parts 1 and 2, so that (if a lies to the

left of b) all the points of 1 lie to the left of all the points of 2, then there exists just one point c which brings about this separation, so that the points of 1 lie between a and c, those of 2 between c and b. This corresponds obviously to the introduction of irrational numbers by means of the Dedekind cut.[66]

From these axioms we can actually derive by logical deduction the whole of projective geometry of space. In particular, we could, of course, promptly introduce coordinates and treat projective geometry analytically.

If we desire to go over to *metric geometry*, we must take into consideration that in projective geometry we have also the notion of the *group of* ∞^{15} *collineations* or projective transformations of space. We know how to characterise, as a subgroup of this, the seven-parameter *principal group* of motions in space whose invariant theory constitutes metric geometry. This group consists of the collineations, which leave unchanged a certain plane, namely, the *infinitely distant plane*, and in that plane a curve of the second degree, namely, the *imaginary spherical circle* (or the *absolute polar system* which represents it). However, we must go a step farther than this, if we wish to get exactly the theorems of elementary geometry. We must separate out from the principal group the six-parameter *subgroup of proper motions* **[174]** (translations and rotations) which, unlike the similarity transformations, leave the distance between two points wholly unchanged. In this way, we shall have the metric geometry of congruencies as our invariant theory. We can derive the motions from the principal group, for example, by setting up the requirement that the "*path curves*" *of a motion are closed* insofar as it leaves only one point fixed.

The plan thus sketched for building up geometry is *theoretically* perhaps the *simplest*, since it operates, at first, for projective geometry, exclusively with linear configurations, and only later adjoins a quadratic configuration, the imaginary spherical circle, when this becomes necessary in order to get metric geometry. To carry this plan through is quite an abstract and tedious matter, however, and it would be appropriate only in a proper lecture course of projective geometry alone. It will suffice after this general exposition, to refer you to that presentation in the literature which is the most readable, namely, to the translation, by Hermann Fleischer of the book by Federigo Enriques, entitled *Vorlesungen über projective Geometrie*.[67]

For general teaching purposes, I prefer another method of developing the subject of geometry, to which I now turn. For simplicity's sake I confine myself to *geometry of the plane*.

1. Development of Plane Geometry with Emphasis upon Motions

We shall take as fundamental notions *point* and *straight line*, and we shall assume for them axioms of *connection*, *order*, *and continuity*. Here again, the theorems of connection contain only the facts of intuition *that through any two points there*

[66] See Vol. I, pp. [36]–[37].
[67] Leipzig, 1903 [2nd German edition 1915]. The title of the original is *Lezioni di geometria proiettiva*, Bologna, 1898; third edition, 1909.

always passes one and only one straight line, while two straight lines can have either *one point or none in common*. Concerning the *order* of the points on a line we shall retain the conditions already indicated above. A careful formulation of the additional axioms of order and of the axioms of continuity will be considered during the course of the investigation.

With this foundation, we shall now avoid the roundabout use of projectivities, and we shall introduce immediately the *group of* ∞^3 *motions in the plane*, in order, through it, to reach our goal, the system of plane analytic geometry. First of all, we must formulate abstractly, in a series of axioms, the properties of these "motions", which we shall assume and use, with respect to our system of points and straight
[175] lines. We shall be guided here, of course, by the vivid conception of motion, which we have had in our experience with rigid bodies. Accordingly, a motion must, in the first place, be a biunique transformation of the points of our space. In particular, it must correlate every point with a point lying in finite space. Moreover, it must carry a straight line over into a straight line, without exception. It is convenient to use again, in general, the word *collineation* for transformations of this kind. To be sure, we do not yet know whether or not there are such collineations, since we are not now in possession of projective geometry, as we were before. Hence we must expressly postulate the existence, at least, of these particular collineations, by means of a *new axiom*. Accordingly, we assume *that there is a group of* ∞^3 *collineations, which we shall call motions*, and whose invariant theory we shall look upon as the geometry of the plane. We must explain more precisely what is meant here by "triply infinite." Given any two points A and A' (see Fig. 104) and two rays a and a' drawn from A and A', respectively. Then there will always be one and only one motion which carries the point A into the point A' and the ray a into the ray a'. Figures, which can be carried into each other by motion are called *congruent*.

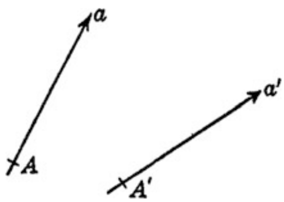

Figure 104

Constructing Affine Geometry from Parallel Translations

However, we shall not yet make use of this entire group of motions, but only of a particular class of motions for which we shall set up some *special postulates*. In fact, there is just one motion which carries a point A into an arbitrary given point A' and the straight line from A to A' (together with this direction) into itself. We call such a motion a *translation*, or, more precisely, a *parallel translation*. *We claim now*

that each such translation carries into itself (*with maintenance of its direction*) *the straight line, which joins any two of its corresponding points B and B′*, and, what is essential, that *the ∞^2 translations of the plane constitute a subgroup of the group of motions*.

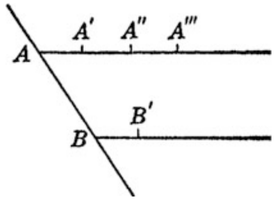

Figure 105

If we perform repeatedly one and the same translation (see Fig. 105), the point A goes over into points A', A'', A''', ... of that half-ray of the straight line AA' which points toward A'. We must assume, as *another postulate*, *that these points ultimately reach or include every point of this half straight line*. By repetition of the inverse transformation we obtain a series of points of the same character on the **[176]** other half straight line. If we think of each translation as capable to be performed continuously, from the initial point to the endpoint, which is what we shall use later, we call the straight line in question the *path curve* of the point A under the translation. Every straight line is thus the path curve of infinitely many points, and for every translation there are ∞^1 path curves, namely, the straight lines, which the translation carries over into themselves.

Now it should be noted that *two different path curves of the same translation cannot intersect*. Otherwise, the point of intersection obviously would result from the translation of two different points, namely, one from each of the two path curves, which is contrary to the character of a translation as a biunique point transformation. We say that all the path curves of one and the same translation are *parallel* to one another. We have thus derived this notion from a property of our motions. At the same time, it is clear that through a given point A there is *certainly o n e parallel to a straight line a*, namely, the path curve of A under a translation along a.

Finally, we must set up a *last axiom* for these translations, *namely*, *that any two translations T', T''* are *interchangeable*, i.e., that the same point B will result

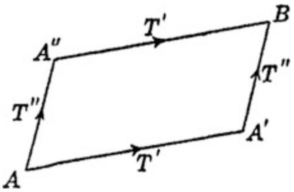

Figure 106

when we subject a definite point A first to the translation T' and then to T'', as when we perform first T'' and then T' (see Fig. 106). Symbolically we may write $T' \cdot T'' = T'' \cdot T'$.

I shall have something to say later regarding the method by which we arrive at such axioms. For the present, let me emphasise that our initial theorems are merely the expression of that, which is familiar to everyone, from the beginning of geometric drawing. Indeed, the first thing that one does is to move a rigid body, ruler or compass or other instrument from one part of the drawing plane to another, in order to transfer quantities. In particular, we perform the operation of translation very often by sliding a triangle, say, along a straight edge (see Fig. 107). Here experience shows again and again that all the points of the triangle describe parallel lines. Our assumptions, which we shall not analyse logically any further, are thus not in the least artificial.

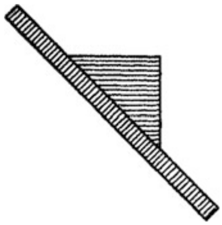

Figure 107

[177] We shall now see how far we can get in analytic geometry with these first notions derived from translations. We cannot talk about rectangular coordinates, of course, since we have nothing yet upon which to base a definition of a right angle. We can, however, introduce *general parallel coordinates*. We draw, through a point O, any two straight lines, which we call *the x-axis and the y-axis*. (See Fig. 108.) We consider the translation T, which carries 0 into an arbitrarily chosen *point* 1 on the x-axis, and we suppose that repetition of the translation T yields the *points* $2, 3, 4, \ldots$ on the x-axis. If we perform, in the same way, the *inverse operation* T^{-1}, so defined as to transfer 1 into 0, the point 0 will go successively into the *points*

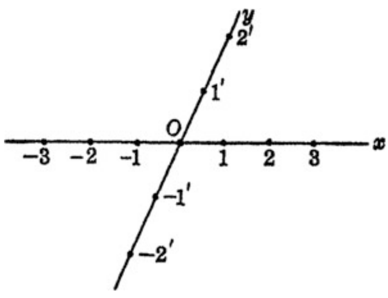

Figure 108

$-1, -2, -3, \ldots$ of the x-axis. We assign to the points thus obtained the positive and negative *integers* $0, 1, 2, \ldots, -1, -2, \ldots$ *as* "*abscissas*" x. To be sure, they will not exhaust all the points on the x-axis, but they will, according to one of our postulates, lie so that every other point will be included between some pair of them.

In similar manner, we start from any translation T' along the y-axis, and, by performing it repeatedly forwards and backwards, we obtain the points $1', 2', 3', \ldots, -1', -2', -3', \ldots$, to which we assign positive and negative *integer* y-*coordinates*. However, we should note here that we cannot set the x- and y-segments, thus determined, into reciprocal relation with each other, since we have not yet introduced the motion (rotation), which would carry the x-axis into the y-axis.

We can now consider the *points on the x-axis with non-integer abscissas*, if we keep fixed the arbitrarily determined unit. We shall discuss first the *rational points*. In order to make the matter clear by an example, we shall seek a translation S along the x-axis, which, if repeated once, would produce the unit translation T. We shall denote as the point $1/2$ that point into which S transfers O, while repeated application of S will yield points with abscissas $3/2, 5/2, \ldots$ In order to establish the existence of such a translation S and of these points, we shall first show that the straight line from $1/2$ to the point $1'$ on the y-axis must be parallel to the line $1\,2'$ (which corresponds to the known construction for bisecting a segment). Indeed, if we consider the translation S (see Fig. 109) of 0 to $1/2$ as made up of the translation **[178]**
T' of 0 to $1'$, followed by the translation S' of $1'$ to $1/2$, then the once repeated translation S, which, by definition, is identical with T, can be replaced, in view of the interchangeableness of two translations, by the once repeated translation T' followed by the once repeated translation S'. But since the first transfers 0 to $2'$, this amounts to saying that two applications of S' transfer $2'$ to 1. Then $2'\,1$ is a path curve of the translation S' and, as such, is parallel to $1'\,\frac{1}{2}$, a path curve of the same translation.

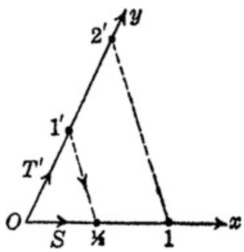

Figure 109

By what precedes, we are already in possession of the points $2'$ and 1, and consequently of the translation S'. Thus the unique constructability, from given elements, of the point $1/2$, as the intersection of the x-axis with the path curve from $1'$ in the translation S', would be assured if we only knew that this path curve really cuts the x-axis. Of course, no one would doubt this, intuitively, but in the framework of our axiomatic deduction we need here a special axiom, the so-called "*betweenness ax-*

iom" for the plane. This axiom states that if a straight line enters a triangle through one side, it must leave it through *another* side – a trivial fact of our space intuition, which requires emphasis as such, because it is logically independent of the other axioms. Completely analogous considerations show, obviously, the existence of *a point for every rational abscissa x*. We can easily infer from our postulates that there are such "rational points" inside of every segment, however small it may be.

In order, now, really to reach all the points, which we actually consider in geometry, we must take into account irrational abscissas. For this purpose we need a new, likewise very obvious axiom, one that is merely a *precise statement* of the *requirements of continuity* mentioned above. *There should be infinitely many other points on the x-axis* (*translations of the axis into itself*), *which have to the rational points the same relations of order and continuity which the irrational numbers have to the rational*. This axiom is the more plausible, in that, conversely, the introduction of irrational numbers came about historically from a consideration of geometric continuity.[68] We have, finally, *all the points of the x-axis brought into biunique correspondence with all the positive and negative real numbers x. An analogous relation can of course be set up for the points of the y-axis*.

[179] Let me remind you that the method thus sketched for constructing a scale on a straight line is a thoroughly natural one. When we make a scale, we do it by sliding a rigid body that has the arbitrary length of the unit (say the distance between the points of the compass) along a ruler, and by subdividing the segments thus obtained.

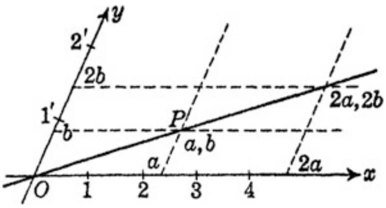

Figure 110

Each translation of the plane along the x-axis can now be characterised by a simple equation, which, for every point x of the x-axis, gives the abscissa of the new position: $x' = x + a$. In other words, the rational or irrational, positive or negative segment a is added to x. Similarly, a translation along the y-axis is described by the equation $y' = y + b$. If we perform both these translations successively (see Fig. 110), in either order, O goes over into a definite point P, since the translations are interchangeable. We say that P has the *abscissa a* and the *ordinate b*. Conversely, to any point P one can assign uniquely two numbers a and b. We need only translate O to P and determine the abscissa and ordinate of the intersections of the new positions of the axes with their original positions. *There is thus established a one to one correspondence between the totality of the points in the plane*

[68] See the discussion in Vol I, pp. [34] sqq.

and the totality of number pairs (a, b), *i.e.*, *we have a complete determination of coordinates in the plane*.

It remains for us to consider how the *equation of the straight line* looks. Let us study first the line from O to P (a, b). Obviously, it must contain all the points which arise through iteration of the translation which transfers O to P, i.e., the points $x = \lambda a, y = \lambda b$, where λ is an integer. Moreover, we see that the points determined by these equations for rational values of λ, and finally for irrational values of λ, also lie on this line, but that then all the points on the line are exhausted. Eliminating λ, we obtain the equation of the line in the form $x : y = a : b$, or $bx - ay = 0$. It follows that every equation of the form $\alpha x + \beta y = 0$ represents a line through O, provided that α and β do not vanish simultaneously. Now any line can be derived from a selected line through O by translation. It follows then, finally, *that all straight* **[180]** *lines are given by all equations of first order*,

$$\alpha x + \beta y + \gamma = 0\,,$$

which, for this reason, are called *linear equations*.

From the fact that the straight line has a linear equation, it follows that a large part of the theorems of geometry can be derived without difficulty by methods of analytic geometry. I cannot go into details here, and I add merely that we can deduce in this way the *whole of affine geometry* and hence also the *whole of projective geometry*. We can get this far simply on the basis of the special postulates concerning the subgroup of ∞^2 translations. I shall lay stress upon only one more fact, which we shall use later. We proved earlier, by means of the theorems of projective geometry, the *theorem of Möbius*, that every *collineation* is a *projective* transformation, i.e., a transformation which is given by a linear fractional or a linear integer substitution of coordinates. Now, according to our first assumptions, all motions were collineations, *under which there corresponds to every finite point likewise a finite point*. On the other hand, however, we possess now the whole of projective geometry, so that, from our standpoint, the theorem of Möbius is also valid. Thus *every motion will be represented necessarily by a linear integer transformation of the parallel coordinates* x *and* y, which were *introduced* above (see p. [183]).

Adding Rotations to Construct Metric Geometry

If we wish now to enter farther into the *metric notions of geometry*, and, in particular, to know about the *angle* between two straight lines and the *distance* between two arbitrary points – thus far we can talk only of the distance between two points on the x- or on the y-axis, we must turn our attention to the *entire group of motions*.

We shall consider, in particular, the motions, which leave a point, say the origin O, unchanged. These are the so-called *rotations* about this point. According to the general postulate concerning the determination of a motion, *there is just one rotation* which transfers a half-ray a through O into an arbitrary half-ray a' through

O (see Fig. 111). These rotations are, in a sense, *dual* to the translations, since they leave a point unchanged, whereas translations carry a straight line into itself. Just as with the translations, we shall think of the rotations as carried out continuously from the initial position on, and we shall talk again of the *path curve*, which each point describes.

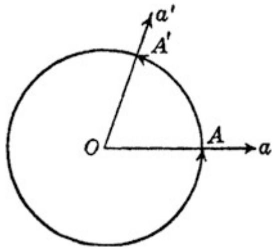

Figure 111

There is, however, one essential difference between rotations and translations,
[181] which we must expressly formulate here as a special postulate. *The half-rays* $a', a'', \ldots$, *which are derived from a by repetition of one and the same rotation about O ultimately coincide with or include every ray through O* (whereas a translation only yields the points of a single ray). In particular, therefore, the continuous rotation of the ray a must ultimately return it to its initial position, whereby each point of a returns to its original position. *The path curves are thus closed lines* which meet each ray through O in just one point A, so that all segments OA are congruent to each other (i.e., can be carried over into one another by a motion). They are what are commonly called *circles with centre O*.

By means of these rotations, we shall now establish a *scale* in the family of rays about O, much as we constructed a scale on the straight line by means of translations. In this also we must make suitable *assumptions* as to *continuity*. I do not need to carry this out in detail and I give only the result, that we associate with every rotation a real number, the *angle of this rotation*, and every real number appears as an angle of rotation. The periodicity of the rotation appears, of course, as a new concept, and it would be natural to select, as a unit, the complete rotation, which carries a ray into itself. As a matter of tradition, however, we select *as unit a quarter of a rotation*, which, when repeated four times, gives a full rotation and whose angle is called *a right angle R*. Each rotation is thus measured by its angle $\omega \cdot R$, where ω may be any real number, but may be restricted, on account of periodicity, to the values from 0 to 4 (see Fig. 112).

In the same way, we can define the angle scale in the family of rays about any other point O_1. But, with the aid of translation, we can *transfer the angle scale of O immediately to* O_1. Indeed, if (see Fig. 113) the rays a_1 and a_1' through O_1 are given, and if T is the translation which transfers O into O_1, then we designate by a_1 and a_1' the rays through O into which the rays a_1 and a_1' go under the reciprocal
[182] translation T^{-1}. If, now, Ω is the rotation about O which transfers a into a', then

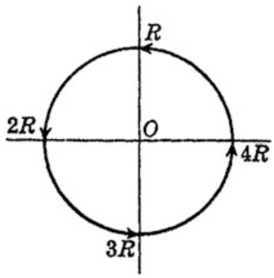

Figure 112

the rotation Ω_1, of a_1 into a_1' about O_1, is given by the succession of T^{-1}, Ω, and T, or, in symbols,

$$\Omega_1 = T^{-1}\Omega T.$$

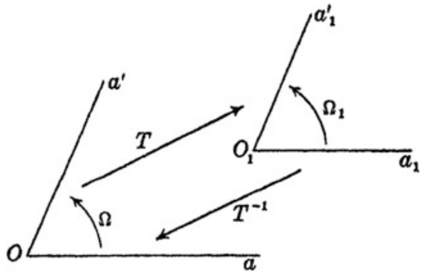

Figure 113

This follows from the fact that the right side represents also a motion, which transfers $O_1 a_1$ into $O_1 a_1'$, and such a motion is uniquely determined. We assign now to Ω_1 the same angle $\omega \cdot R$, which Ω has by the above definition. If we have a second rotation, Ω', in the family O, there will correspond to it, in the family O_1, the rotation

$$\Omega_1' = T^{-1}\Omega' T\,,$$

and the combination of Ω_1 and Ω_1' is

$$\Omega_1 \Omega_1' = T^{-1}\Omega T T^{-1}\Omega' T = T^{-1}\left(\Omega\Omega'\right)T\,,$$

which corresponds to the composition of Ω and Ω'. It follows that our transfer actually gives the same scale at O_1 that would arise through repetition of the original procedure.

There is a theorem in *Euclid*, which is omitted from most of our elementary textbooks, *that all right angles are congruent*. Of course every boy will look upon this theorem as self-evident, and I think that it should be ignored in the schools,

since the pupils do not understand what it means. However, its content is identical with the result of the preceding discussion, namely, that equal angles, which are defined by rotations at different points, can be brought into coincidence by motions, i.e., that they are congruent.

Now that we have given a general definition of angle, we shall define the *distance between two arbitrary points*. Thus far we have been able to compare distances only on one and the same line by means of translation. If a distance r is laid off on the x-axis, say, from O, we can transfer it (see Fig. 114) by rotation about O, to any line a' through O. Then we can transfer to a' the scale of length on the x-axis and then also, by translation, to any straight line parallel to a', and thus to any straight line whatever. *We can*, *then*, *actually measure the distance between any two points by joining them by a straight line and transferring to it*, *in the way indicated*, *the scale on the x-axis*. In particular, we shall think of the scale initially chosen for the y-axis as having been derived thus from the one on the x-axis.

Using the new notion of rotation, we shall now complete our apparatus for *ana-*
[183] *lytic geometry*. In doing this, we shall use, as we now may, the special *rectangular coordinates x and y*, instead of general parallel coordinates (see Fig. 115).

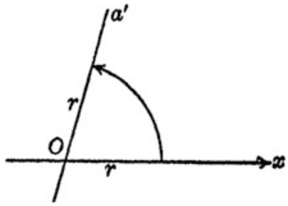

Figure 114

We know already (p. [180]) that every motion is given by a linear substitution in x and y:

$$x' = (a_1x + b_1y + c_1) : N\,,$$
$$y' = (a_2x + b_2y + c_2) : N\,.$$

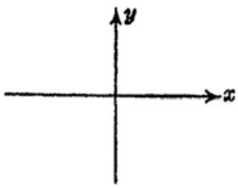

Figure 115

Since this transfers each finite point into another finite point, the denominator N must be constant and may be set equal to 1. If we consider in particular a rotation

about O, then $c_1 = c_2 = 0$, and we have

(1) $$x' = a_1 x + b_1 y\,, \quad y' = a_2 x + b_2 y\,.$$

For the special rotation through a *right angle*, we can state at once the form of the equations. Since we have rectangular coordinates, the x-axis is transferred into the y-axis, and the y-axis into the negative x-axis, so that we have

(2) $$x' = -y\,, \quad y' = x\,.$$

The question as to the determination of the formulas of rotation is now reduced to the following purely analytic problem. *We seek a simply infinite group of substitutions of the form* (1) *which shall include the substitution* (2) *and such that, if ω is a real parameter, every substitution of the group, speaking generally, arises from* (2) *by an ω-times iteration.* For a rational fractional value of ω, say p/q, this expression means, of course, that the substitution repeated q times gives the substitution (1) iterated p times, while an irrational value of ω is to be approximated by rational values, according to our assumptions regarding continuity. It must be understood clearly that we may presuppose no geometric knowledge whatever, especially concerning the formulas of rotation of a rectangular coordinate system; however, we may and we shall use all of our knowledge of analysis without any scruples. The structure, which we thus erect will certainly not be immediately usable for teaching in school, but it does assume a very elegant and simple form.

I shall start with the remark that the rotation (2), by the use of *complex numbers*, can be expressed by the one formula

(2′) $$x' + iy' = i(x + iy)\,.$$

From this form we see that the result of two successive applications of the substitution is represented by the relation $x' + iy' = i^2(x + iy)$. This is an equation of **[184]**
the same form, where the factor i^2 has taken the place of the factor i. Similarly an ω-times iteration, in the foregoing sense, produces the factor i^ω for each real ω. We have, therefore, as the *analytic representation of the rotation of the plane about O through the angle $\omega \cdot R$*, the formula

(3) $$x' + iy' = i^\omega(x + iy)\,.$$

In order to carry out this line of thought with precision, we must assume from analysis a complete knowledge of the exponential function e^z, and also a complete knowledge of the trigonometric functions, which satisfy Euler's formula

$$e^{iz} = \cos z + i \sin z\,.$$

In writing down this relation we do not need to have, at present, even a suspicion of its geometric significance.

We know also the number π, by means of the formula $e^{i\pi} = -1$, and we may write

$$i = e^{\frac{i\pi}{2}}\,.$$

By i^{ω} we understand here the value uniquely defined by the formula

$$i^{\omega} = e^{\omega\frac{i\pi}{2}} = \cos\frac{\omega\pi}{2} + i\,\sin\frac{\omega\pi}{2}\,.$$

If we substitute this value in (3), and separate the real and the imaginary parts, we have

$$\text{(4)}\qquad \begin{cases} x' = \cos\frac{\omega\pi}{2}\cdot x - \sin\frac{\omega\pi}{2}\cdot y \\ y' = \sin\frac{\omega\pi}{2}\cdot x + \cos\frac{\omega\pi}{2}\cdot y\,, \end{cases}$$

which is, in more elementary analytic symbols, the desired representation of the rotation group.

With this result, it is natural *to choose*, *as the unit*, *not the right angle*, *but the angle* $\pi/2$. We shall call this the natural angle scale, as we speak of the natural logarithm, to indicate that these notions are based upon the nature of things, although their full appreciation requires deeper insight. In this natural scale we write simply ω instead of $\omega\pi/2$, and we have, as formulas of rotation, instead of (4), the well-known equations

$$\text{(5)}\qquad \begin{cases} x' = \cos\omega\cdot x - \sin\omega\cdot y\,, \\ y' = \sin\omega\cdot x + \cos\omega\cdot y\,. \end{cases}$$

[185] We must now examine these formulas to see what geometric truths they contain. These will turn out to be all those elementary theorems, which are usually relied upon, in order to set up the formulas (5).

1. Let us start with a consideration of the point on the x-axis, at a distance r from the origin: $x = r$, $y = 0$. If we turn it through the angle ω, the formulas (5) give as coordinates of its new position

$$\text{(6)}\qquad x = r\,\cos\omega\,,\quad y = r\,\sin\omega\,,$$

where, for brevity, the accents have been omitted from the new coordinates. If, to fix ideas, we take $\omega < \pi/2$ and consider the right triangle (see Fig. 116) formed by the radius vector r, the abscissa x, and the ordinate y of the point (x, y), then the formulas (6) exhibit the connection between the sides and the angle ω. From the relation $\cos^2\omega + \sin^2\omega = 1$, which follows from the analytic definition of these functions, we find at once from (6)

$$\text{(6a)}\qquad x^2 + y^2 = r^2\,.$$

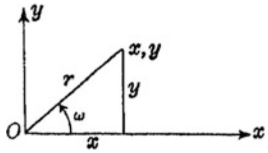

Figure 116

This is the *Pythagorean theorem*, which we thus obtain as *a result of our assumptions concerning motions in the plane*. Moreover, we can write (6) in the form

$$\text{(6b)} \qquad \cos\omega = \frac{x}{r}, \quad \sin\omega = \frac{y}{r}.$$

We obtain thus the elementary trigonometric significance of our angle functions, which is the exact form in which they are usually defined: The cosine and the sine are the ratios of the adjacent side and the opposite side, respectively, to the hypotenuse.

Definitive Establishment of the Terms for Distance and Angle

2. It is now easy to state the *general analytic expressions for the fundamental notions distance and angle*, if we bring the given elements, points or straight lines, through translation and rotation, into the special position just considered. For *the distance between two points* (x_1, y_1) *and* (x_2, y_2), we have

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

To obtain this result, it is merely necessary to transfer the point (2) to the origin by a translation, whereupon, by the translation formulas (p. [179]), the differences $x_1 - x_2$, $y_1 - y_2$ become the new coordinates of the point (1), and (6a) gives at once our expression for r. In the same way, we obtain from (6b) for the *angle* ω *between* **[186]**
two straight lines whose equations are $\alpha_1 x + \beta_1 y + \delta_1 = 0$, $\alpha_2 x + \beta_2 y + \delta_2 = 0$, the formula

$$\cos\omega = \frac{\alpha_1\alpha_2 + \beta_1\beta_2}{\sqrt{\alpha_1^2 + \beta_1^2}\sqrt{\alpha_2^2 + \beta_2^2}}, \quad \sin\omega = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\sqrt{\alpha_1^2 + \beta_1^2}\sqrt{\alpha_2^2 + \beta_2^2}}.$$

I hardly need to give the details of the proof.

Classifying the General Terms Area and Length of Curves

3. Finally we have still to discuss the *notion of area*, of which we have not made the slightest use, thus far, in our development of geometry. Nevertheless, this notion is present in the naive space consciousness of every person, even if in more or less inexact form. Every peasant knows what it means to say that a piece of land has an area of so many acres. When we have succeeded, then, in laying completely the foundations of geometry – and we have actually done just that – without using this fundamental notion, it behooves us to add it now as a supplement to the system, i.e., to express it in terms of coordinates.

We must begin with a simple *geometric discussion*, such as the one given in Euclid or the one given in the elementary presentations. If we have a rectangle with sides A and B, we define its *area* to be the *product* AB. If we combine two rectangles, or any two figures of known area, we define the area of the resulting figure as the *sum* of the two areas. If we remove from a rectangle, or from another figure, a smaller piece lying entirely within it, the remainder has for area the *difference* of the given areas. (See Fig. 117.)

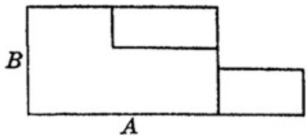

Figure 117

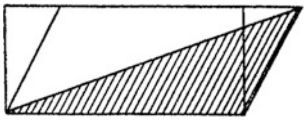

Figure 118

With these conventions, we can proceed at once to the *area of a parallelogram*. This figure arises from a rectangle of equal base and height by taking away a triangle and adding a congruent one. (See Fig. 118.) Hence its area is equal to that of the rectangle, i.e., to the *product of base and altitude*, A diagonal divides the parallelogram into two congruent triangles, each of which has for area, therefore, half that of the parallelogram: *The area of a triangle is half the product of base and altitude*.

[187] If we apply this to a triangle with sides r_1, r_2, and the included angle ω, the altitude upon r_1 is $r_2 \sin\omega$; hence the area is

$$\Delta = \frac{r_1 r_2 \sin\omega}{2}.$$

If we place one vertex of this triangle (see Fig. 119) at the origin and call the coordinates of the other two vertices (x_1, y_1) and (x_2, y_2), then this formula, with the aid of the above expressions for distance and for angle, can be written in the form

$$\Delta = \frac{x_1 y_2 - x_2 y_1}{2}.$$

It is easy to show that rotations of the coordinate system leaves this expression Δ unchanged, so that it really supplies a "geometric concept." In order to have invariance under translation, and so under all motions, however, we must transform also the third vertex, i.e., we must set up the formula for the area of a triangle with vertices at three arbitrary points (x_1, y_1), (x_2, y_2), (x_3, y_3). We obtain in this way:

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

which is, indeed, the formula with which we began this lecture course (see p. [3]). It is easy to show that, if triangles are combined or subdivided, their areas, defined as above, are added or subtracted. The proof, as we saw earlier, depends upon simple determinant relations.

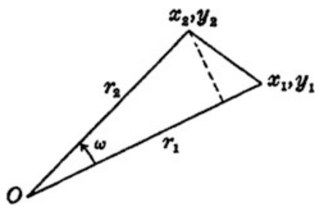

Figure 119

The addition of the idea of area to our system of analytic geometry is thus completed, and we have, at the same time, gained something, which is not contained in the naive conception: Area has become a quantity affected with a *sign*. I discussed (see pp. [4] sqq.) at the beginning of this lecture course the great advantage thus gained with respect to free operating with the formulas and their universal validity, in contrast with the naive notion of area as an absolute quantity.

4. Another example of a concept, which occurs with more or less precision in the naive space intuition, which we must add only now as a supplement to our system of geometry, is the *notion of an (arbitrary) curve*. Every person believes that he knows what a curve is until he has learned so much mathematics that the countless possible abnormalities confuse him. Good orientation in this field can be found **[188]**
in the *Enzyklopädie report* by *Hans von Mangoldt* entitled *Die Begriffe "Linie" und "Fläche"* (III A B 2). We shall not bother here with details but we shall state simply that, for us, a curve is the totality of points whose coordinates are continuous

functions ϕ and χ of a parameter t, which are differentiable as many times as may be necessary:

$$x = \phi(t) , \quad y = \chi(t) .$$

Proceeding in this manner, we can develop immediately, in the frame of our analytic geometry, all of the notions and theorems, which are comprised usually under the name *infinitesimal geometry*, the notions of *length of arc*, *area* of curved *surfaces*, *curvature*, *evolute*, etc. The fundamental idea is always that we think of the curve as the limit of an inscribed rectilinear polygon (see Fig. 120). If the coordinates of two neighbouring points are (x, y) and $(x+dx, y+dy)$, then it follows at once from the pythagorean formula that the *length of arc is*:

$$\int \sqrt{dx^2 + dy^2} ,$$

and it follows also, in the same way, from the formula for the area of a triangle with vertex at O, that the *area of the sector* between the curve and two radius vectors is given by the formula (see p. [11]):

$$\frac{1}{2} \int (x\, dy - y\, dx) .$$

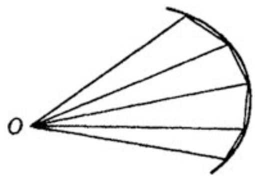

Figure 120

With this I leave our first foundation of geometry, which was characterised by our placing in the foreground the existence and structure of the three-parameter group of motions and then introducing coordinates, in order thereafter to make our inferences entirely within the field of arithmetic. There is a second method of founding geometry, one which is, in a sense, opposed to this. It leads likewise directly to metric geometry and it has always played an important role. We shall now turn our attention to it.

2. Another Foundation of Metric Geometry – the Role of the Parallel Axiom

The contrast to the first foundation consists in this, that now the *concept of motion is consistently avoided*, or, at most, brought in as an afterthought. The fact that this

arrangement was preferred in ancient times, as it frequently still is, was due, in part, to philosophical considerations, which I should at least mention. It was feared that motion would bring into geometry an element foreign to it, namely, the notion of time. When an attempt was made to justify a preference for motion by the marked intuition of the idea of a rigid body, the objection was raised that this idea in itself **[189]** had no precise comprehensible meaning. On the contrary, it was held that this idea could have meaning for us only if we already possessed the notion of distance. The empiricist can always reply, of course, that the abstract idea of distance can actually be inferred only from the presence of "sufficiently" rigid bodies. However, let me now indicate briefly the principal thoughts of this second foundation of geometry.

Distance, Angle, Congruence as Fundamental Concepts

1. We begin, just as before, with the introduction of *points* and *straight lines*, and with the *theorems* concerning their *connection*, *order*, and *continuity*.

2. Besides these – and this is new here – we assume the new fundamental notions, on one hand, of the *distance between two points* (segment) and, on the other hand, the *angle between two straight lines*; and we set up axioms concerning them which state, in substance, that *segments and angles can be measured by numbers in the customary manner*.

3. Here *the first congruence theorem* appears as the following characteristic axiom, which really replaces the axioms of the group of motions: *If two triangles have two sides and the included angle respectively equal*, *they are congruent*, i.e., they are equal in all their parts. In our earlier system, this was a provable theorem, for we can find a motion, which (see Fig. 121) brings the side $A'B'$ into coincidence with AB. Then $A'C'$ necessarily falls along AC, because of this assumption, and the triangles coincide throughout. But if we do not include motions among the fundamental notions, i.e., if we may not use them, there is no possibility of proving this theorem, and we must of necessity postulate it as a new axiom.

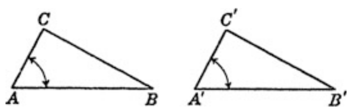

Figure 121

4. For continuing this foundation, the procedure is precisely opposite to that in our first foundation, as you know. Elementary geometry teaching does this consistently, adhering essentially to the procedure of Euclid, of whom I shall have something to say later. It is customary first to prove the Pythagorean theorem, and then to introduce the trigonometric functions cosine and sine, from their meaning in the theory of triangles. From this beginning, the same analytic apparatus is finally derived as the one before.

Parallel Axiom and Theory of Parallels (Non-Euclidean Geometry)

5. In this process it becomes necessary to set up another axiom, which is very important, concerning the *theory of parallels*. In our first foundation, parallelism was one of the first fundamental notions, which appeared immediately upon consideration of translations. Straight lines were called parallel if they were path curves of
[190] the same translation. Here it is entirely different.

Parallelism is not among the fundamental notions considered thus far, and we must now discuss it. Indeed, if we have a straight line g (see Fig. 122) and a point O outside it, we join O with a point P of g and let P move out along g through the positions P', P'', ... In other words, we consider the succession of points $P, P', P'', \ldots$, or the succession of straight lines $OP, OP', OP'', \ldots$, whereby there is no concept of motion implied, in the earlier sense. The ray OP, under these circumstances, will reach a limiting position when P moves off to infinity, and we *call this limiting straight line a parallel to g through O*. It does not appear at all necessary that OP should approach the *same* limiting position when P goes to infinity in both directions, so that the *abstract possibility* arises of the existence of two *different parallels to g through O*.

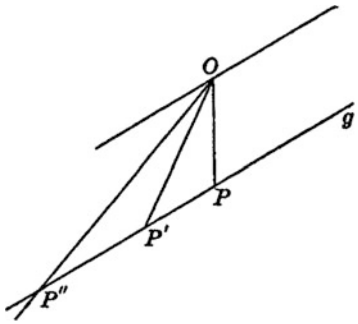

Figure 122

In our present foundation, therefore, it is a *new axiom* if, according to our habitual intuition, we postulate that the two limiting positions should coincide, i.e., that *there is only* o n e *parallel through a given point to a given line*. This is the famous *parallel axiom*, concerning which there has been so much dispute these many hundreds of years. It is also called *Euclid's axiom*, since he expressly formulated it as a postulate.

I should report you *something of the history of this axiom*. Through many years men used their best efforts in the attempt to *prove* the axiom, i.e., to show that it was a consequence of the other axioms, but always in vain. Of course, these attempts have not all been abandoned, even today. For although science can advance ever so far, there will always be people who think that they know better and who ignore the assured results of exact research. The fact is that mathematics has long since advanced, beyond these futile attempts, to fruitful new investigations and to

positive results. As early as during the eighteenth century, there was raised the following *characteristic question*, *suggestive of new possibilities*: Is it not possible to set up a logically consistent system of geometry, free from contradictions, in which the parallel axiom is set aside, and in which the existence of *two* different limiting straight lines in the sense discussed above, i.e., of two different parallels to g through O, is admitted?

At the beginning of the nineteenth century, this question could be answered affirmatively. It was Gauß who first discovered the existence of *a* "*non-Euclidean*" *geometry*, which is the name that he gave to such a geometric system. His *Nachlass* shows that he certainly knew this already exactly, in 1816. To be sure, the **[191]** notes in which he discussed these things were found only much later and were not printed until 1900 in volume 8 of his collected works.[69] Gauß himself had published nothing about this great discovery, beyond a few occasional remarks. The jurist Ferdinand Karl Schweikart, about 1818, independently of Gauß, constructed a non-Euclidean geometry, which he called astral geometry, but he likewise did not publish his results. They became known first through a letter to Gauß which was found in the latter's *Nachlass*. The first publications on non-Euclidean geometry came from the Russian, Nikolai J. Lobatschefsky (1828), and the Hungarian, Janos Bolyai de Bolya, the younger (1832),[70] both of whom had got these results independently of each other and were in possession of proofs by 1826 and 1823, respectively. In the course of the century, these things have come into the general possession of mathematicians through numerous works, so that today, indeed, every person of general culture has heard of the existence of a non-Euclidean geometry, even though only an expert can attain a clear understanding of it.

In the early part of the second half of the nineteenth century Riemann gave an essentially new direction to these problems. His work appeared in 1854 in his *Habilitation* lecture entitled *Über die Hypothesen welche der Geometrie zugrunde liegen*.[71] Riemann remarked that all the preceding investigations were based on the *assumption that the straight line was of infinite length*, which was certainly very natural and obvious. He asked what would happen if we should give up this assumption, that is, if we should allow the straight line to return into itself, as does the great circle on the sphere. We are confronted here with the *difference between the infinity and the unboundedness of space*, which can best be seen, perhaps, in two-dimensional space. The surface of the sphere and the ordinary plane are both unbounded, but only the second is infinite, whereas the first is of finite extent. Riemann assumes, in fact, that *space is only unbounded and not infinite*. Then the straight line on which the points lie will be a closed curve similar to a circle. If **[192]**

[69] Leipzig, 1900. This volume was edited by Paul Stäckel.

[70] Translated into German in *Urkunden zur Geschichte der nichteuklidischen Geometrie* by Friedrich Engel and Paul Stäckel: Part I (Lobatschefsky) by Engel (Leipzig, 1898). [Part 2 (W. and J. Bolyai) by Stäckel, Leipzig, 1913.) See also *Urkundensammlung zur Vorgeschichte der nichteuklidischen Geometrie* by Stäckel and Engel, Leipzig, 1895.

[71] Published in vol. 13 of *Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen* = *Gesammelte mathematische Werke*, 2nd ed., p. 272 et seq. (Leipzig, 1892). [New edition by Hermann Weyl, 3rd edition. Berlin: Springer, 1923.]

now we let a point P move, as before, in a definite direction, farther and farther on a straight line g, it will ultimately return to its original position. The ray OP of our former discussion will not have a limiting position and *there will be no parallel to g through O*. Thus there appears with Riemann a *second kind of non-Euclidean* geometry ("N.G. II"), in contrast with the non-Euclidean geometry of Gauß, Bolyai, and Lobatschefsky ("N.G. I").

This seems at first paradoxical, but the mathematician notices here, at once, a *relation to the ordinary theory of quadratic equations*, which points the way to an understanding of the matter. Indeed, a quadratic equation has either two different real roots, or none at all (both being imaginary), or finally, as a transition case, one real root counted twice. This is entirely analogous to the two different real parallels in N.G. I, to the absence of real parallels in N.G. II, and finally to the transition case of one parallel counted in two ways, as the same limiting position, in Euclidean geometry.

Philosophical Importance of Non-Euclidean Geometry

Before I enter more carefully upon the discussion of non-Euclidean geometry, I shall touch, at least briefly, upon its great *significance from the philosophical side*, by virtue of which it has always aroused tremendous interest with the philosophers, but has also often been flatly rejected.

Above all, this discipline informs us about the *character of geometric axioms looked at from the standpoint of pure logic*. Indeed, from the existence of non-Euclidean geometry, we can conclude at once that the Euclidean axiom is not a consequence of the preceding fundamental notions and theorems, nor are we under any other logical compulsion to accept it. For if we retain all the other axioms but replace this one by a contrary assumption, we are not led to a contradiction, but we obtain, rather, non-Euclidean geometry, as a logical structure, which is just as correct as is Euclidean geometry. *Details of our spatial perception, such as those described in the parallel axiom, are thus certainly not a purely logical necessity*.

The question arises, now, whether or not, perhaps by means of *sensory intuition*, we can decide as to the correctness of the parallel axiom; upon this also non-Euclidean geometry provides revealing information. *In fact, it is certainly not true that immediate sensory intuition teaches us the existence of just o n e parallel*. For, our spatial perception is decidedly *not absolutely exact*. As in every other region of sense perception, so here, we can no longer recognise as distinct quantities (segments, angles, etc.) whose difference lies below a certain limit, the so-called *threshold of perception*. Thus if we draw, in particular, through the point O, two
[193] lines very close to one another (see Fig. 123), certainly we can no longer distinguish between them if we make the angle between them small enough, say $1''$, or, if one will, $\frac{1}{1000}''$, or even still smaller. Thus it would be difficult to decide, by immediate sensory intuition, whether there is really just one parallel to g through O, or two which are separated from each other by such a small angle. We sense this still more

distinctly if we think of O as very far away from g, say as far away as Sirius, or a million times that far. With such distances, sensory intuition loses completely the keenness which we otherwise expect of it, and we should certainly no longer be able to determine visually whether the limiting position of the rotating ray provided one or two parallels to the given straight line g.

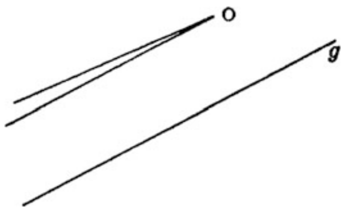

Figure 123

Now this situation actually fits into the non-Euclidean geometry of the first kind just as well as it does into Euclidean geometry. As we shall soon see, when we look into the mathematical formulas, there is an arbitrary constant involved. By a suitable choice of this constant, we can make the angle between the two parallels to g arbitrarily small if the point O is moderately distant from g, and this angle becomes appreciably large only when O is sufficiently remote from g. *In view of the fact that our space intuition is adapted only to a limited part of space, and then only with a limited degree of accuracy, it can obviously be satisfied by a non-Euclidean geometry of the first kind, N.G. I, as closely as we please.*

But a similar thing is true also for N.G. II (Riemannian non-Euclidean geometry). It is only necessary to become conscious that the infinite length of the straight line cannot be an inference from our sensory intuition. We can follow any straight line only in a finite part of space; consequently it cannot contradict our space experience if we say that the line has a length that is enormously great but still finite, perhaps a million or more times the distance to Sirius. Imagination can conjure up arbitrarily large numbers, which exceed every possibility of immediate intuition. *In accord with these considerations, we can represent the situation in any limited part of space with any desired degree of accuracy by means of N.G. II (a Riemannian non-Euclidean geometry), for such a geometry which again implies an arbitrary constant.*

The logical and intuitive facts here touched upon, as they present themselves from the standpoint of mathematics, run counter in high degree to that conception of space which many philosophers connect with the name Kant, and according to which all theorems of mathematics must have absolute validity. This explains why non-Euclidean geometry, since its introduction into philosophical circles, has attracted so much attention and aroused so much opposition. **[194]**

Integration of Non-Euclidean Geometry into the Projective System

If we turn now to a *proper mathematical treatment of non-Euclidean geometry*, we shall do best to choose the *path through projective geometry*. That is the derivation, which I gave in 1871 in volume 4 of the *Mathematische Annalen*.[72]

We think of projective geometry as developed from the fundamental notions point, line, plane, and their axioms of connection, order, and continuity, independently of any metric, as I indicated briefly in the beginning of the discussion of the foundations of geometry (pp. [172]–[173]). In particular, we introduce point coordinates x, y, z, or homogeneous coordinates $\xi : \eta : \zeta : \tau$, and also plane coordinates α, β, γ, δ, so that the mutual incidence of point and plane is given by the bilinear equation

$$\alpha\xi + \beta\eta + \gamma\zeta + \delta\tau = 0\,.$$

Upon this foundation we have already set up *ordinary Euclidean geometry*, by means of the theory of invariants and Cayley's principle, by adjoining the special quadratic form written in plane coordinates

$$\Phi_0 = \alpha^2 + \beta^2 + \gamma^2\,,$$

which, set equal to zero, represents the imaginary spherical circle. The angle between the two planes

$$\omega = \arccos \frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2}{\sqrt{\left(\alpha_1^2 + \beta_1^2 + \gamma_1^2\right)\left(\alpha_2^2 + \beta_2^2 + \gamma_2^2\right)}}\,,$$

and the distance between two points

$$r = \frac{\sqrt{(\xi_1\tau_2 - \xi_2\tau_1)^2 + (\eta_1\tau_2 - \eta_2\tau_1)^2 + (\zeta_1\tau_2 - \zeta_2\tau_1)^2}}{\tau_1\tau_2}$$

were then, as we showed (pp. [168] sqq.), simple simultaneous invariants of the given figure (the two planes or the two points) and the form Φ_0.

We are going to try to set up *non-Euclidean geometry* in a similar way. Instead of the imaginary spherical circle $\alpha^2 + \beta^2 + \gamma^2 = 0$, we take *another quadratic form,* which is "near" the preceding one, namely:

$$\Phi = \alpha^2 + \beta^2 + \gamma^2 - \varepsilon \cdot \delta^2\,.$$

Here ε is a parameter which can be chosen arbitrarily small, and for $\varepsilon = 0$, we have $\Phi = \Phi_0$. Our form is so chosen that for *positive* e we get non-Euclidean geometry of the first kind; for negative ε, arises N.G. II; while for $\varepsilon = 0$, we get the preceding

[72] *Über die sogenannte nichteuklidische Geometrie*, pp. 573 sqq. = [F. Klein, *Gesammelte mathematische Abhandlungen*, vol. 1, pp. 254 sqq.].

formulas for ordinary Euclidean geometry. It is essential in the setting up of this **[195]**
form Φ that its *determinant*

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon \end{vmatrix} = -\varepsilon$$

is, in general, different from zero. The determinant vanishes only in the special case $\varepsilon = 0$, i.e., when $\Phi = 0$ represents the imaginary spherical circle. Our assumption then amounts to this, *that we replace the quadratic form whose determinant vanishes by a quadratic form whose determinant is positive or negative* (*but arbitrarily small in absolute value*).

We shall obtain the metric quantities for our non-Euclidean geometries by constructing, from the general form Φ and from the figure consisting either of two planes or of two points, invariants entirely analogous to those, which represent the Euclidean quantities for the special form $\Phi_0 = \alpha^2 + \beta^2 + \gamma^2$. This is nothing else than the notion of Cayley,[73] developed in 1859, *that one can define a system of measurement just as well with respect to any quadratic surface* (*e.g., the surface* $\Phi = 0$) *as with respect to the spherical circle*. In view of the limited space to which this digression is confined, it will be expedient to set down the analytic formulas in advance. In this way the situation can be most quickly outlined with precision, and every shadow of mystery avoided. Of course, this presentation can lead to a full understanding of the material only if it is afterwards worked through carefully from the geometric side, as you will find it done in my article, already mentioned, in volume 4 of the *Mathematische Annalen*.

If we first consider two planes, it seems natural to set up the expression for the "*measure of the angle between them with respect to the surface* $\Phi = 0$" by generalising the preceding expression for the angle. Just as there, we construct, from the values of the form Φ and of its polar form, the formula

$$\omega = \arccos \frac{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 - \varepsilon\delta_1\delta_2}{\sqrt{\alpha_1^2 + \beta_1^2 + \gamma_1^2 - \varepsilon\delta_1^2}\sqrt{\alpha_2^2 + \beta_2^2 + \gamma_2^2 - \varepsilon\delta_2^2}}.$$

In this way we obtain an expression which is obviously invariant, which for $\varepsilon = 0$ actually goes over into the formula for angle of Euclidean geometry.

It is not so immediately clear how one can transform the *expression for the distance between two points* into our metric. In fact, the difficulty in the change lies in the fact that we now have a form whose determinant does not vanish, instead of the form Φ_0, whose determinant vanishes, which characterised Euclidean metric. **[196]**
However, we can discover how to set up the expression for distance if we *proceed exactly dualistically to the definition of the angle just given*. In this way, we are certain to get an invariant. We set up first, then, the equation of the surface $\Phi = 0$

[73] In the *Sixth Memoir upon Quantities*, already cited (p. 145).

in point coordinates. We get its left side $f(\xi, \eta, \zeta, \tau)$, as you know, by bordering with point coordinates the determinant Δ of Φ:

$$f = \begin{vmatrix} 1 & 0 & 0 & 0 & \xi \\ 0 & 1 & 0 & 0 & \eta \\ 0 & 0 & 1 & 0 & \zeta \\ 0 & 0 & 0 & -\varepsilon & \tau \\ \xi & \eta & \zeta & \tau & 0 \end{vmatrix} = \varepsilon\left(\xi^2 + \eta^2 + \zeta^2\right) - \tau^2 .$$

We now transfer the expression for ω by writing the quotient of the polar form of f divided by the product of the square roots of the values of f formed for the points 1 and 2, and then taking the arc cosine:

$$r = k \arccos \frac{\varepsilon\left(\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2\right) - \tau_1\tau_2}{\sqrt{\varepsilon\left(\xi_1^2 + \eta_1^2 + \zeta_1^2\right) - \tau_1^2}\sqrt{\varepsilon\left(\xi_2^2 + \eta_2^2 + \zeta_2^2\right) - \tau_2^2}} ,$$

The factor k which we have inserted permits us to make an arbitrary segment equal to unity, as we are in the habit of doing. Moreover, this will become necessary when we go over to Euclidean geometry. We must think of *k as real when ε is negative and as pure imaginary when e is positive*, in order that r shall be real for all real points or at least for a certain subregion of all real points (when $\varepsilon > 0$), which then make the real substratum of non-Euclidean geometry.

We have now reached a general definition of distance. It remains, only, to show *that, for $\varepsilon = 0$, it leads to the customary expression of Euclidean geometry*. This is not so easy here as it was before for the angle ω, for if one sets $\varepsilon = 0$ outright, the quotient is 1, and r/k is equal to zero, to within an undetermined additive multiple of 2π. In spite of this somewhat paradoxical result, we can nevertheless obtain finally the Euclidean expression by means of a certain device. To this end, it is convenient to transform the defining equation for r by means of the equation $\arccos\alpha = \arcsin\sqrt{1-\alpha^2}$ to a common denominator, we find that the value of r is

$$k \cdot \arcsin \cdot \sqrt{\frac{\left\{\varepsilon\left(\xi_1^2+\eta_1^2+\zeta_1^2\right)-\tau_1^2\right\}\left\{\varepsilon\left(\xi_2^2+\eta_2^2+\zeta_2^2\right)-\tau_2^2\right\}-\left\{\varepsilon(\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2)-\tau_1\tau_2\right\}}{\left\{\varepsilon\left(\xi_1^2+\eta_1^2+\zeta_1^2\right)-\tau_1^2\right\}\left\{\varepsilon\left(\xi_2^2+\eta_2^2+\zeta_2^2\right)-\tau_2^2\right\}}} .$$

[197] We can now easily transform the numerator. Indeed, using a known determinant relation, the value of f (i.e., the determinant Δ of the form Φ, once bordered) for the point 1, multiplied by the same determinant for the point 2, minus the polar form taken for points 1 and 2, can be shown to be equal to the product of the determinant Δ itself by the determinant Δ bordered twice with the coordinates of 1 and 2, that

is, equal to the product

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 & \xi_1 & \xi_2 \\ 0 & 1 & 0 & 0 & \eta_1 & \eta_2 \\ 0 & 0 & 1 & 0 & \zeta_1 & \zeta_2 \\ 0 & 0 & 0 & -\varepsilon & \tau_1 & \tau_2 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 & 0 & 0 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 & 0 & 0 \end{vmatrix}.$$

Performing this multiplication, we find

$$-\varepsilon \cdot \Big\{(\xi_1\tau_2 - \xi_2\tau_1)^2 + (\eta_1\tau_2 - \eta_2\tau_1)^2 + (\zeta_1\tau_2 - \zeta_2\tau_1)^2 \\ - \varepsilon\,(\eta_1\zeta_2 - \eta_2\zeta_1)^2 - \varepsilon\,(\zeta_1\xi_2 - \zeta_2\xi_1)^2 - \varepsilon\,(\xi_1\eta_2 - \xi_2\eta_1)^2\Big\}\,.$$

Anyone who is not skilful in calculating with determinants can show by direct transformation that this expression is identical with the numerator in the preceding expression for r. If we insert this expression in the formula for r and put $\varepsilon = 0$, we get, of course, just as in the first form,

$$\frac{r}{k} = \arcsin 0 = 0\,,$$

because of the factor $\sqrt{-\varepsilon}$. But if we do not allow ε to become azero, but only to become very small, the arc sine is, as a first approximation, equal to the sine. We can neglect, in the numerator, the three squares, each multiplied by ε, and, in the denominator, that term in each factor which is multiplied by ε. There remains, as a first approximation,

$$r = k \cdot \sqrt{-\varepsilon}\,\frac{\sqrt{(\xi_1\tau_2 - \xi_2\tau_1)^2 + (\eta_1\tau_2 - \eta_2\tau_1)^2 + (\zeta_1\tau_2 - \zeta_2\tau_1)^2}}{\tau_1 \cdot \tau_2}\,.$$

We come now to the device mentioned above. *During the passage to the limit*, $\lim \varepsilon = 0$, *we do not assign to k a fixed value*, *but we let it become infinite in such a way that* $\lim(k \cdot \sqrt{-\varepsilon}) = 1$. For this purpose we must, of course, let k pass through pure imaginary or through real values according as ε approaches zero through positive or through negative values. Thus it is evident *that the expression for distance in Euclidean geometry* (*p*. [194]) *actually does emerge from this passage to the limit*.

If we think our way into the geometric significance of the form f as well as the significance of the expressions, which have been only analytically put down here, it turns out *that we actually have*, *for* $\varepsilon > 0$, *non-Euclidean geometry of the first kind*, **[198]**
for $\varepsilon < 0$, *that of the second kind*, *and for* $\varepsilon = 0$, *of course*, *Euclidean geometry*. To be sure, I cannot give the whole argument here. For that I must refer you to my article in volume 4 of the *Mathematische Annalen*.[74] At that time I proposed for

[74] [Attention is again drawn to *Einführung in die nichteuklidische Geometrie* by F. Klein (edited by W. Rosemann), which is about to appear as a revision of the earlier mimeographed volume of Klein's lectures on non-Euclidean geometry.] [Translator's note: The new edition was published, in fact, in 1928. The original version is of 1892.]

these three geometries the names *hyperbolic*, *elliptic*, and *parabolic*, since the existence of two real, two imaginary, or two coincident parallels corresponds precisely to the behaviour of the asymptotes of these three conic sections, respectively. You will find these names frequently in the literature.

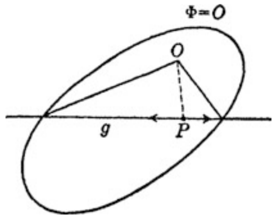

Figure 124

I should like to show in greater detail, by an example, what form the theory of parallels takes from the expression for distance. For this purpose I choose *hyperbolic geometry* in the *plane*. We must then set the third coordinate equal to zero. Our quadratic form becomes $\Phi = \alpha^2 + \beta^2 - \varepsilon\delta^2$ which, equated to zero, represents a *real* conic section, which we can think of as an ellipse, since $\varepsilon > 0$. The distance formula takes the form

$$r = k \arccos \frac{\varepsilon\left(\xi_1\xi_2 + \eta_1\eta_2\right) - \tau_1\tau_2}{\sqrt{\varepsilon\left(\xi_1^2 + \eta_1^2\right) - \tau_1^2}\sqrt{\varepsilon\left(\xi_2^2 + \eta_2^2\right) - \tau_2^2}},$$

where k is pure imaginary. It yields, as it is easy to see, real values for points which lie inside the real conic section, where we mean by inside points the totality of the points in a plane from which no real tangents to the conic section can be drawn. *Hence the field of operations of the real hyperbolic geometry consists exclusively of these interior points and of the straight lines which lie in this interior.* The points on the conic section (see Fig. 124) itself represent the *infinite* region. For, the formula yields the value ∞ for the distance of each point 1 from a point 2 on the conic section [for which $\varepsilon\left(\xi_2^2 + \eta_2^2\right) - \tau_2^2 = 0$]. Thus there are, in this sense, two *infinitely distant points* in hyperbolic geometry on every straight line, namely its intersections with the conic section $\Phi = 0$, but there is only *one* on each half-ray *a*. If we have a line g, and a point O not lying on it, then the parallels through O, in the sense of our earlier definition (p. [176]), as the limiting positions of the lines joining O with a point, which moves along g to infinity, are the lines
[199] joining O with the intersections of g with the conic section. *There are*, *in fact*, *two parallels*, *essentially different from each other*, each of which belongs to one of the two directions on g.

Let me make one more brief remark, which concerns a *comparison with our first foundation of Euclidean geometry*. We started there with the *group of motions*. That was the totality of collineations, which left the metric relations unchanged. But there are likewise such collineations in non-Euclidean geometry. A general

homogeneous equation of the second degree has ten terms and therefore nine essential constants. In the most general space collineation there are fifteen arbitrary parameters, so that there is a *six-fold infinity of collineations, which transform a given quadratic form*, *e.g.*, *our Φ form*, *into itself*. Indeed, this is the condition that the metric relations, which we have introduced should remain unchanged. *Hence there is also in each non-Euclidean geometry a six-fold infinite group of* "*motions*" *which leave ω and r unchanged*. For geometry in the *plane* the number of parameters would reduce, as before, to three.

We can, therefore, develop each non-Euclidean geometry also by starting from the *existence of a group of motions*. It remains only to point out how it came about that our earlier development led us exclusively to *Euclidean* geometry. The reason was, of course, that we selected from among the motions the special two-parameter (in space it would be a three-parameter) *subgroup of so-called parallel translations*, which had only straight lines as path curves. *There are no such subgroups in any non-Euclidean geometry*, *and since we postulated their existence at the beginning*, *we excluded non-Euclidean geometry once and for all and retained only Euclidean geometry*.

Let me conclude this special discussion of non-Euclidean geometry with a few *general advisory statements*, as I may call them.

1. Whereas I reported earlier that, from the side of philosophy, non-Euclidean geometry had frequently not been received with full understanding, I must emphasise that it is today quite generally recognised in the *science of mathematics*. In fact, for many purposes, e.g., in the modern theory of functions and in the theory of groups, it is used as a very convenient means for making clear visually relations that are arithmetically complicated.

2. *Every teacher certainly should know something of non-Euclidean geometry*. Thus, it forms one of the few parts of mathematics, which, at least in scattered catch-words, became known to a larger public, so that any teacher may be asked about it at any moment. In physics there are, of course, far more such things, **[200]** which are on every tongue and about which, therefore, every teacher should be informed. Indeed, almost every discovery in physics belongs into this category. Imagine a teacher of physics who is unable to say anything about Röntgen rays, or about radium. A teacher of mathematics who could give no answer to questions about non-Euclidean geometry would not make a much better impression.

3. On the other hand, I should like to advise emphatically against bringing non-Euclidean geometry into *regular school teaching* (i.e., beyond occasional suggestions, upon inquiry by interested pupils), as enthusiasts are always recommending. Let us be satisfied if the preceding advice is followed and if the pupils learn really to understand *Euclidean* geometry. After all, it is in order for the teacher to know a little more than the average pupil.

General Remarks About Modern Geometric Axiomatics

I should like to consider briefly the *further development of modem science,* which has been occasioned by non-Euclidean geometry. A good starting point was made from one of its results, namely, that the Euclidean parallel axiom was logically independent of the other axioms of geometry (see p. [192]). This *stimulated the study of the other geometric axioms as to their mutual logical dependence or independence*. From this arose the *modern theory of geometric axioms*, which in its procedure follows closely the path, which the older investigation had disclosed. In it, we determine what parts of geometry can be set up without using certain axioms, and whether or not, by assuming the opposite of a given axiom, we can also secure a system free from contradictions, that is, a so-called "*pseudo-geometry*."

As the most important work belonging here, I should mention Hilbert's *Grundlagen der Geometrie*.[75] Its *chief aim* as compared with earlier investigations is to *establish*, *in the manner indicated*, *the significance of the axioms of continuity*. To accomplish this, it is of course necessary, above all, to arrange the system of geometric axioms so that the *theorems on continuity* come at the end, whereas for us they have thus far stood at the beginning. Thus we were unable, in our development of non-Euclidean geometry, to make use of the first arrangement of the axioms (pp. [174] sqq.), which put the notion of parallels at the head. To the contrary, we were obliged to create a system of axioms in which the greater part of the discussion said nothing about parallels, and in which the parallel axiom was added at the
[201] end. Setting aside the essential departure thus indicated, Hilbert's system of axioms accords, in the main, with the construction of elementary geometry as also used in our second foundation (pp. [188] sqq.).

With this basis, Hilbert inquired *in how far geometry can be developed without using the axioms of continuity. He includes in the treatment also the* "*pseudo-geometries*," *in which all the other geometric axioms are valid*, *excepting only the axioms of continuity*. Such geometries consist essentially of those facts, which are concerned with the one-to-one correspondence between the points of a straight line and the ordinary real numbers (their abscissas). (See p. [173] and p. [177]) Of course, I cannot give the details of the argumentation in Hilbert's investigations or the interesting results which he obtained concerning the logical connection between certain geometric theorems and axioms. With these few explanatory remarks, I leave it to you to read all this in Hilbert's own writings. Let me recall, however, that his *non-archimedean geometry*, which we discussed in the first volume of this lecture course[76] belongs here. This is, indeed, such a pseudo-geometry in which that axiom of continuity which was formerly named after Archimedes, but which now often bears the name of Eudoxus, is no longer satisfied, i.e., in which the abscissas of two different points may differ by an "actually infinitely small quantity," of which no finite multiple is equal to an ordinary finite real number.

[75] 5th edition, Leipzig and Berlin, 1922
[76] See Vol. I, pp. [235]–[236].

I do not wish to conclude these brief remarks on the modern theory of axioms without saying a few words on the important question concerning the *true nature of geometric axioms and theorems*. Of course, this takes me out of the strict field of mathematics into that of philosophy and the theory of knowledge. I have already emphasised one thing about which most people today are in reasonable agreement. That is that we are concerned here with the *leading concepts and statements, which one must of necessity put into the front rank of geometry in order to be able to realise mathematical proofs derived from them by pure logic*. This statement does not answer the question as to the *real source of these leading concepts and theorems*. There is the old point of view that they are the *intuitive possession of every person*, and that they are of such obvious simplicity that no one could question them. This view, however, was shaken, in large measure, by the discovery of non-Euclidean geometry; for here it is clearly shown (see pp. [191] sqq.) that space intuition and logic by no means lead compellingly to the Euclidean parallel axiom. To the contrary, we saw that, with an assumption, which contradicts the parallel axiom, we come to a logically closed geometric system, which represents actual perceptual relations with any desired degree of approximation. However, it may well be claimed that this parallel axiom is the assumption, which permits the *simplest* representation **[202]** of space relations. Thus it is true in general that *fundamental concepts and axioms are not immediately facts of intuition*, *but are appropriately selected idealisations of these facts*. The precise notion of a point, for example, does not exist in our immediate sensory intuition, but is only a fictitious limit, which, with our mental pictures of a small bit of shrinking space, we can approach without ever reaching.

In contrast with this, one finds frequently now, on the part of persons who are interested only in the logical side of things and not in the side of intuition or of the general theory of knowledge, the opinion that *the axioms are only arbitrary statements which we set up at pleasure and the fundamental concepts*, *likewise*, *are only arbitrary symbols for things with which we wish to operate*. The truth about such a view is, of course, that *within pure logic* there is no room for these statements and concepts, and that they must therefore be supplied or suggested from other sources – precisely through the influence of intuition. Many authors express themselves much more one-sidedly, however, so that in recent years, in the modem theory of axioms, we have frequently found ourselves led in the direction of that philosophy which has long been called *nominalism*. Here interest in things themselves and their properties is entirely lost. What is discussed is the way things are named, and the logical scheme according to which one operates with the names. For example, it is said that we *call* the aggregate of three coordinates a point, "without thinking of any particular object," and we agree "arbitrarily" upon certain statements, which shall hold for these points. In such a discussion, we may set up axioms arbitrarily, and without limit, provided only that the laws of logic are satisfied and, above all, that no contradictions appear in the completed structure of statements. For one, I cannot share this point of view. I regard it, rather, as the death of all science. *The axioms of geometry are – according to my way of thinking – not arbitrary*, *but sensible statements*, *which are*, *in general*, *induced by space intuition and are determined as to their precise content by expediency*.

As a counterpart to the philosophical digressions to which we have repeatedly been led in the foregoing pages, I should like to give some *account of the history of geometry*, in particular of the development of views concerning its foundations. In contrast with similar considerations, which we repeatedly gave last winter in the fields of algebra, arithmetic, and analysis, we notice, at the outset, a great difference. These other disciplines, in their modern form, really have a history of only a few centuries. They had their start when men began to calculate with decimal fractions and letters, in round numbers about the year 1500. *Geometry*, however, *as an in-*
[203] *dependent discipline has a history reaching far back into Greek antiquity*. Indeed, it had even then reached such a high stage of development that for a long period, reaching almost to the present time, men looked upon Greek geometry as a prototype of a completed science. At the same time, the famous *Elements* (*στοιχεῖα*) *of Euclid*, by far the most significant systematic textbook to survive, was looked upon as the whole of Greek mathematics. There is, indeed, hardly another book, which, for so long, maintained such a place in its field of science. Even today, every mathematician must come to terms with Euclid. To him, therefore, we shall devote the last section of the present chapter.

3. Euclid's Elements

Let me first put before you the edition of this work prepared by J. L. Heiberg[77] of Copenhagen, which is the best from a philological standpoint. In it, the Latin translation of the original Greek text is added, which is also very helpful for those who have not studied Greek. Indeed, Euclid's Greek differs widely, especially in the technical terms, from the Greek taught in the schools. As literature to serve as an introduction to Euclid, I should recommend Zeuthen's *Geschichte der Mathematik im Altertum und Mittelalter*[78] and Max Simon's *Euklid und die* 6 *planimetrischen Bücher.*[79] You will find your way into the subject if you read first Simon, then Zeuthen's more general discussion, and then the text of Heiberg, but the latter should be read by all means carefully and with a critical mistrust of each translation.

Very little is known of Euclid personally. We know only that he lived *in Alexandria about* 300 B.C. However, we are informed about the *general scientific activity* that existed in Alexandria. After the founding of Alexander's world empire, there arose gradually the need for collecting and bringing into a unified scientific system, everything that the past centuries had created, so that there developed in Alexandria a system of teaching which corresponded closely to certain aspects of our university teaching of today. But the *collection and arrangement of the material at hand*

[77] *Euclid's Opera Omnia*, Books I-V, *Elementa*, Leipzig, 1883–1888.

[78] Copenhagen, 1896.

[79] Leipzig, 1901 = *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XL [See also the annotations of T. L. Heath in his English translation of the Heiberg text: *The Thirteen Books of Euclid's Elements*, 3 vols., Cambridge, 1908.]

took precedence over the free onward drive of scientific research, so that a certain tendency to pedantry manifested itself in this whole activity.

Critical Remarks About the Historical Importance and Scientific Significance of the Elements

Before we go over to a detailed analysis of the *Elements*, let me make some general remarks about *the place in history and the scientific importance of Euclid*, *or* **[204]**
rather of Euclid's Elements. Although a complete picture of Euclid's personality would require the consideration of his numerous lesser writings, I am nevertheless justified in discussing here only the one great work; for this alone has achieved the remarkable commanding position, which, from our standpoint, urgently, demands criticism.

As a justification for this criticism, I offer the remark that the underlying reason for the erroneous appraisal of Euclid's *Elements* is a *mistaken belief as to the Greek spirit*, which was widespread for a long time, and which indeed still persists. It was believed that Greek culture confined itself to relatively few fields, but that it wrought in these fields with such complete mastery that its achievements must remain a paradigm for all time supreme and unattainable. The fact is, however, that modern philological science has long since shown this view to be untenable. It has taught us, rather, that the Greeks, as no other people, busied themselves, with the greatest possible versatility, in *all* fields of human culture. Just as their accomplishments in every field were certainly admirable, for their era, so certainly they failed in many things to get beyond what we now consider the very beginnings. In no field can it be said that they attained the all-time summit of human achievement.

As to *mathematics*, in particular, this overestimate – or should I say underestimate? – of Greek culture and science found expression in the dogma that the Greeks had given very substantial attention to geometry and had set up there a system that could not be surpassed. This belief had led, in particular, to an outright *cult of Euclid's Elements*, in which it was claimed that such a system had been completely realised. In opposition to this old and outworn belief, I make the assertion that although *the Greeks worked fruitfully*, *not only in geometry*, *but also in the most varied fields of mathematics*, *nevertheless we today have gone beyond them everywhere and certainly also in geometry*.

Let me now explain this assertion more in detail and try to justify it. In writing his *Elements*, Euclid wished by no means to compile a cyclopedia of the accumulated geometric knowledge of his time; otherwise he would not have disregarded entire portions of geometry, which were certainly known in his day. I need mention only the *theory of conic sections and of higher curves,* which the Greeks had already begun to treat extensively,[80] although we owe its full development to Apollonius **[205]**
(about 200 B.C.). Moreover, the *Elements* were to be merely an *introduction* to

[80] Euclid had himself written a work on conic sections, which has not survived.

the study of geometry, and therefore to mathematics itself. Hence it seems they were intended for a particular purpose. They were to treat mathematics in the way considered necessary, in the sense of the platonic school, as a *preparation for philosophical studies in general*. With this in mind, we see why emphasis was placed upon working out the logical connections and upon setting forth geometry as a closed system, while all practical applications were laid aside. In favour of this system, however, Euclid certainly passed over an entire part of the theoretical knowledge of his time, which was not far enough developed to fit into his needs.

We can best obtain a correct impression of the *limited character of the subjects of Euclid's Elements*, compared with the range of Greek mathematics as a whole, if we use for comparison the individuality and the achievement of the *most eminent of Greek mathematicians*, *Archimedes*, who lived shortly after Euclid, in Syracuse, about 250 B.C. I shall mention only a few especially interesting and distinguishing facts.

1. In marked contrast to the spirit dominant in Euclid's *Elements*, Archimedes shows a strongly developed *sense for numerical calculation*. Indeed, one of his greatest feats, to mention only one definite example, was the calculation of the number π by approximating to the circle with regular polygons. Among other results, he derived the approximation 22/7 for π. Euclid shows no trace of interest for such numerical values. Instead, we find in Euclid the fact that two circles are to each other as the squares of their radii, or that two circumferences are to each other as the radii themselves; but the calculation of the proportionality factor, this number π, is not even attempted.

2. Characteristic of Archimedes was his far-reaching *interest in applications*. It is well known that he discovered the fundamental principle of hydrostatics, and that he took an active part in the defence of Syracuse, by constructing effective machines. How little thought Euclid gave to applications, on the contrary, appears clearly from the fact that he does not once mention even the simplest drawing instruments – the ruler and compass. He merely postulates, in the abstract, that one can draw a straight line through two points, or a circle about a point, without devoting a single word to *how* one does it. Here Euclid is doubtless under the influence of
[206] the notion, which prevailed in certain ancient schools of philosophy, that practical application of a science was something inferior, artisan-like. Unfortunately this view persists in many places today, and there are still always university teachers who cannot be too scornful of any concern with applications, as being ignoble. The arrogance of such views should be vigorously combatted. We should value equally highly every admirable performance, whether in the theoretical or in the practical field, and we should allow each individual to concern himself with those things to which he feels most strongly inclined. In this way, any person will show himself the more versatile, the more talents he possesses. The most eminent mathematicians, as Archimedes, Newton, Gauß, have always uniformly included both theory and applications.

3. Finally, another difference attracts particular attention. Archimedes was a *great researcher and pioneer*, who, in every one of his works, made advances in knowledge. Euclid's *Elements*, however, are concerned merely with the *collec-*

tion and systematisation of knowledge already at hand. That is the reason for the difference in the *form of presentation*, to which I drew your attention last semester when I was talking more generally.[81] In this connection, there is an especially characteristic manuscript[82] of Archimedes, which was discovered in 1906 (mentioned in Part I), in which he confides to a scientific friend his most recent investigations on the volumes of space figures. His presentation resembles closely our present method of teaching. He proceeds *genetically*, first indicating the train of thought, and by no means using the rigid arrangement of hypothesis, proof, conclusion, which characterises the Euclidean *Elements*. Moreover, it was known before this new discovery, that the Greeks had, besides this crystallised "Euclidean" presentation of a systematised discipline, also a free genetic form, which was used, not only by the researcher, but also by the teacher in his instruction. Presumably Euclid also employed this method in his other works as well as in his teaching. Indeed, there was in Alexandria at that time an analogy of our present-day mimeographed volumes of lecture notes, called *hypomnemata*, i.e., loose-leaf reproductions of oral presentations.

This will suffice as a comparison of the *Elements* with the whole range of Greek mathematics. As a conclusion of this discussion, I shall show, by means of a few simple examples, how far *modern mathematics* has advanced beyond that of the Greeks. One of the important differences is that the Greeks possessed *no independent* arithmetic or *analysis*, neither decimal fractions, which lighten numerical **[207]** calculation, nor general use of letters in reckoning. Both of these, as I showed in my lecture course last winter, are inventions of early modern times, during the Renaissance. As a substitute, the Greeks had only a *calculus in geometric form*, in which operations were performed constructively with segments or other geometric quantities, instead of with numbers, a process much more cumbersome than is our arithmetic. Coupled with this also is the fact that the Greeks did not have *negative and imaginary numbers*, which are really what give facility to our arithmetic and analysis. Consequently they lacked the generality of method, which permits the inclusion in a formula of all possible cases. A most tedious distinguishing of cases played the greatest role with them. This lack is often very noticeable in geometry, whereas today, by employing analytic aids, as we have actually done in this lecture course, we can easily achieve complete generality, and we can avoid all distinction of cases. These few indications will suffice here. You will be able, from your own knowledge, to give many other instances of the advance of modern mathematics as compared with that of the Greeks.

[81] See Part I, p. [80].

[82] See Heiberg und Zeuthen, *Eine neue Schrift des Archimedes*, Leipzig, 1907. Bibliotheca Mathematics, 3rd series, vol. 7, p. 321 et seq. [See also the edition of Archimedes by T. L. Heath, which was translated into German by F. Kliem (Berlin, 1914); the handwriting is reproduced there, p. 413 et seq.]

The Content of the 13 Books of Euclid

After this general criticism on Euclid's *Elements*, we can turn to a special analysis. Let me begin with a *brief survey of the* "*thirteen books*," i.e., chapters, of which they consist.[83]

Books 1–6 are devoted to *planimetry*. The *first four books contain general considerations about fundamental geometric forms*, such as segment, angle, area, etc., and the *theory of simple geometric figures* (triangles, parallelograms, circles, regular polygons, etc.), in the manner in which they are usually given today. In this connection, there is given (Book 2) an *elementary arithmetic and algebra of geometric quantities* in which – to give but one example – the product $a \cdot b$ of two segments a, b is represented as a rectangle. If we wish to add two such products $a \cdot b$ and $c \cdot d$, which we can carry out at once arithmetically, it is necessary, in order to represent the product as a single rectangle again, to transform the two rectangles $a \cdot b$ and $c \cdot d$ into rectangles with equal bases.

Book 5 goes much deeper, in that it introduces the *geometric equivalent of the general positive real number*. This is the ratio a/b of any two segments a, b, which
[208] Euclid calls logos ($\lambda \acute{o} \gamma o \varsigma$). I referred to this last semester, in my general discussion of irrational numbers.[84] The essential keynote of this development is the *definition of the equality of two ratios* a/b and c/d. This definition must be perfectly general, and must hold, therefore, when a/b is, in our sense, irrational, i.e., when the segments a and b are (as Euclid says) *asymmetroi*, i.e., without a common measure, or, as it was translated later, *incommensurable*. Euclid proceeds as follows: He takes any two integers m and n and compares, as to size, the two segments $m \cdot a$ and $n \cdot b$ on the one hand, and $m \cdot c$ and $n \cdot d$, on the other. There must obtain one of the three relations

$$m \cdot a \;\substack{>\\=\\<}\; n \cdot b \quad \text{or} \quad m \cdot c \;\substack{>\\=\\<}\; n \cdot d \,.$$

If, then, for arbitrary values of m, n, *the same sign always holds in both cases, we say that* $a/b = c/d$. This corresponds completely, in fact, to the famous *cut process* by means of which *Dedekind* introduces irrational numbers.

Euclid now proceeds with the consideration as to how one can reckon with such equations between ratios, and he develops his well-known *theory of proportion*, i.e., a geometric theory of all possible algebraic transformations of equations of the type $a/b = c/d$. Euclid uses for a proportion the word *analogia* by which he means that the *logos* of the two pairs of magnitudes is the same. You see how far the word has drifted away today from its original meaning. There are places in mathematics, however, where the word retains its old meaning. We still speak in trigonometry of *Napier's analogies*, because these have to do with certain proportions. To be sure, few persons seem to know the real meaning of this name.

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[83] [One speaks also of Books 14 and 15 of the *Elements* (vol. 5 of Heiberg's edition); but these two books are not by Euclid. The first is rather ascribed to Hypsikles; the second is ascribed to Damaskios.]

[84] See Vol. I, pp. [35]–[36].

The *theory of proportion* is a characteristic example of the persistence with which the Euclidean tradition maintains itself in mathematics teaching. Even today, this theory is taught in many – perhaps, indeed, in most – of the schools, as a *special chapter of geometry*, although it is included completely, in substance, in our modern arithmetic, and has therefore been taught twice before this – once during the study of the proportion, and again in the beginnings of reckoning with letters. Why the same thing should appear a third time, and in especially mysterious geometric clothing, is truly hard to understand for the pupil. The tendency to do so must be quite incomprehensible to the students. Of course, the only reason is that men still **[209]** cling to the old Euclidean curriculum, although, indeed, the sensible purpose which Euclid had in the theory of proportion – to create a substitute for the arithmetic which he lacked – is for us utterly useless.

This criticism of the present-day treatment of the theory of proportion does not refer, of course, to the *scientific importance of the fifth book of Euclid*. That is, of course, great, because there was given here, for the first time – speaking in modern terms - *the rigorous basis for calculation with irrational numbers*, based upon precise definitions. We observe clearly here that Euclid's *Elements* were, and are, by no means a school textbook, as has been so often erroneously assumed. The *Elements* presuppose, rather, a mature reader capable of scientific thinking.

I must mention the tradition that this fifth book was not written by Euclid himself, but by *Eudoxus of Knidos* (circa 350 B.C.). In fact, the *Elements* are looked upon, not as a unified work, written in one piece, but as having been put together out of different older parts.

However this may be, in any case, all of the information as to the authors is clouded with the greatest uncertainty, since there is absolutely nothing extant, in the nature of historical notes, by Euclid or by any of his contemporaries. The above tradition goes back to Proclus Diadochus, a commentator on Euclid who lived about 450 A.D., that is, more than 700 years after Euclid. Even though, for various reasons, the assertion of Proclus may have a certain essential probability, still we should be as little inclined to admit it as absolutely reliable evidence as we should be to accept a theory promulgated today as to the authority of a work compiled around 1200 A.D.

Proceeding with the contents of the *Elements*, we find in Book 6 the *theory of similar figures*, where the principal aid used is the doctrine of proportion.

In Books 7, 8, and 9, the *theory of integers* is treated, *partly in geometric form*. We find here, for proportions with integers, i.e., *for reckoning with rational fractions*, a theory which is entirely independent of the developments of Book 5. Although rational fractions are merely a special kind of real numbers, no reference of any sort is made to the more general theory. It is therefore difficult to believe that the two presentations are by the same author. Of the contents of these books, I should like to mention only two things, both of which are now used in the theory of numbers. One of these is the *Euclidean algorithm* for finding the *greatest common* **[210]** *divisor* of two integers a and b, which Euclid represents by segments. In modern terms, it consists in dividing a by b, then b by the remainder, and so on according

to the scheme

$$a = m \cdot b + r_1\,, \quad b = m_1 \cdot r_1 + r_2\,, \quad r_1 = m_2 \cdot r_2 + r_3\,, \ldots$$

Finally, after a finite number of steps, the division will be exact. The last remainder is the divisor sought. Secondly, one finds in Euclid the well-known simple proof of the *existence of infinitely many prime numbers*, which I gave in my lecture course last winter.[85]

In Book 10, which is especially tedious and hard to understand on account of the geometric form of expression, there is a *geometric classification of irrationalities that are expressible as square roots*, such as were to be used later in geometric constructions.

Not until in Book 11 do we find the beginnings of *stereometry*. You observe that Euclid is no "fusionist." He sets stereometry as far apart from planimetry as possible, whereas we consider it desirable today, in the sense of our oft-mentioned "striving toward fusion," to develop spatial perception as a whole as early as possible, and consequently to accustom the pupil from the beginning to three-dimensional figures, rather than to restrict artificially his first instruction to the plane.

In Book 12 there appear again *general considerations about irrational quantities*, which were necessary *for finding the volume of a pyramid* and of other bodies. Here we find a veiled application of the notion of a limit, in the so-called "proof by exhaustion," by means of which proportions between irrational numbers are rigorously deduced. This method is used first in proving the planimetric theorem that two circles are to each other as the squares of their radii, and it is by means of this example that I shall explain briefly the underlying conception of the method. Any circle can be increasingly approximated by an inscribed n-gon and also by a circumscribed n-gon of an increasing number of sides. It can, in a sense, be *exhausted*, in that the areas of the polygons differ arbitrarily little from the area of the circle. If, then, the proportion did not obtain, one could easily bring about a contradiction
[211] of the fact that every inscribed polygon is smaller than the circle, and that every circumscribed is larger than the circle.[86] (See Fig. 125.)

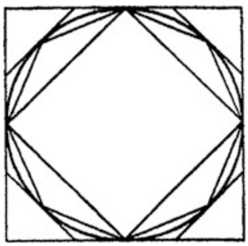

Figure 125

[85] See Vol. I, pp. [43]–[44].

[86] About the relation between the exhaustion method and the modern notion of limit, see vol. I, pp. [225]–[226].

Finally, Book 13 contains the *theory of the regular bodies*, and using the results collected in Book 10, culminates in the proof that one can construct all these bodies, i.e., the lengths of their sides, with ruler and compass. This final result corresponds to the interest, which the Greek philosophers always showed in the regular bodies.

Having given this general survey of the contents, let us turn our attention, in some detail, to those chapters of Euclid which treat of the *foundations of geometry*. The ideal purpose, which Euclid had in mind was obviously the *logical derivation of all geometric theorems from a set of premises completely laid down in advance*. The historical significance of the *Elements* rests mainly, without doubt, on the creation (or transmission) of this ideal. But Euclid did not, by any means, really reach his high goal. Indeed, modern science has gained deeper knowledge, in precisely the fundamental notions of geometry, and has found obscurities in Euclid. Nevertheless, tradition is so strong that Euclid's presentation is widely thought of today, especially in England, as the unexcelled pattern for the foundation of geometry. Men mistake the historical importance of the work for absolute and permanent importance. It is only natural, in view of this over-valuation of Euclid's *Elements*, that I should, in the following discussion, lay emphasis upon the negative side, upon those points in which Euclid's presentation no longer meets our requirements.

A special difficulty arises, in every such criticism of Euclid, in the *uncertainty of the text*. Much of it is attested by Proclus, who is our oldest source. The oldest manuscripts, which we possess are from the ninth century A.D., i.e., they are 1200 years younger than Euclid! Furthermore, these various manuscripts differ greatly, and often precisely in the fundamental parts on which so much depends. Then, too, there is the tradition of Latin and Arabian translators and commentators, in whose works there are many important divergences, due to the efforts to clarify the text. The production of a trustworthy text of the elements is thus an exceedingly complicated philological problem, upon which an amazing amount of acumen has been **[212]**
expended. We must be satisfied with the fact that what is gained by such philological work is, at best, the *most probable text*, but that it *cannot be the true original text*. It by no means follows that what we infer from many different statements, as the most probable course of events, agrees in *all* points with actuality. It is generally admitted that Heiberg's text stands at the summit of modern philological science, and we non-philologists cannot do better than to base our arguments upon it, although we must not forget that it is by no means necessarily identical with the original text. Hence, if we find shortcomings and contradictions in this text, we must always be in doubt as to whether they should be ascribed to Euclid, or whether they slipped in during transmission.

The Foundation of Geometry in Euclid's Elements

And now, coming to the point, let us first inquire how, in Book 1, the *foundations of geometry are laid*. Euclid places at the head three groups of propositions which he calls ὅροι (*definitiones*), αἰτήματα (*postulata*), and κοιναὶ ἔννοιαι (*communes animi conceptiones*) which we may render in German perhaps by *Erklärungen*, *Forderun-*

gen, *und Grundsätze*.[87] For the last group we usually employ, with Proclus, the word *axioms*, which nowadays has extended its meaning to include that of the postulates.

In order first to understand the contents of the *definitions*, let us recall how we started earlier with our foundation of geometry. We said that we could not define certain things, such as point, straight line, plane, but that we must look upon them as fundamental concepts familiar to everybody, and that we should state precisely only such of their properties as we wished to use. With that understanding, we were able to construct geometry, up to the point of producing the system of coordinates (x, y, z) of analytic geometry. Only after that did we consider the general notion of a curve, by thinking of x, y, and z as continuous functions of a parameter t. At that time, I indicated that this would include bizarre degenerations, such as curves, which completely cover a surface, etc.

Euclid did not have this spirit of cautiousness, or of strategic retreat. He begins with the "definition" of all sorts of geometric concepts, such as *point*, *line*, *straight line*, *surface*, *plane*, *angle*, *circle*, etc. The first "definition" runs: *A point is that which has no part*. We are hardly able to recognise this as a proper definition, since a point is by no means determined by this property alone. Again, we read: *A line is length without breadth*. Here, indeed, even the correctness of the statement is
[213] doubtful, if one recognises the general notion of curve, mentioned above, of which Euclid, of course, knew nothing. Then, thirdly, *a straight line is "defined" as a line, which lies evenly with respect to its points*. The meaning of this statement is wholly obscure; all sorts of meaning can be attached to it. It might mean that the line has the same direction everywhere, in which case direction must be admitted as a fundamental notion familiar to everyone. We might also interpret it by saying that a straight line, if realised as a rigid rod, always coincides with itself under certain motions in space, namely, under rotation around itself as an axis or under translation along it. This view of Euclid's "definition" would, to be sure, presuppose the notion of motion; whether Euclid intended that is a disputed question to which we shall return. In any event, it has not been possible to find an unambiguous interpretation for Euclid's definition of the straight line, and likewise for many of his other definitions, which I cannot consider here in detail.

We come now to the postulates, of which five are given in the Heiberg edition. The first three of these require that *it shall be possible*:

(a) *To draw a straight line from one point to another*;
(b) *to prolong indefinitely a limited straight line*;
(c) *to draw a circle with a given centre so as to pass through a given point*;

I shall withhold the fourth, temporarily, and pass on to the fifth, the so-called *parallel postulate*:

(d) *If two straight lines make with a third straight line*, *and on the same side of it*, *interior angles whose sum is less than a flat angle*, *the two lines cut each other*, *if they are sufficiently prolonged toward that side*. (See Fig. 126.)

[87] [In non-technical English, we may call these *explanations*, *agreements*, *and fundamental statements*; in technical terms, *definitions*, *postulates*, and *axioms*. – HEDRICK & NOBLE.]

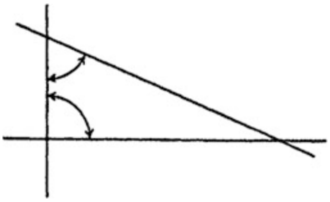

Figure 126

These postulates state the possibility of certain constructions, or the existence of certain geometric figures, of which Euclid makes use later. But there are a considerable number of similar existence-postulates in geometry, which he also uses and which cannot be deduced logically from those that he does state. I shall mention, as one example, the theorem that two circles intersect if each passes through the centre of the other (see Fig. 127). It would be easy to state many other similar theorems. Hence we must say that the *Euclidean system of postulates is certainly deficient*.

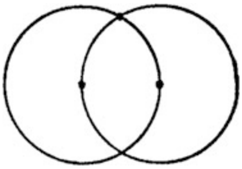

Figure 127

Let us now consider the *fourth postulate*:

(e) *All right angles are equal.*

There has been much dispute as to what this postulate means, and why it appears where it does. Involved with this is the question as to whether or not Euclid uses the concept of motion. If we consistently put at the beginning the notion of the motion of figures as rigid bodies, as we did in our first foundation of geometry, then this postulate follows as a necessary logical consequence (see p. [182]), and it would therefore be superfluous here, even if Euclid otherwise had this point of view. In all these fundamental theorems of Euclid, however, there is nowhere any explicit mention of motion, so that many interpreters assume that this fourth postulate is to serve precisely to *introduce the idea of motion*, though all would admit, to be sure, that the idea would still be in incomplete form. **[214]**

On the other hand, most of the commentators on Euclid think that one of the essential tendencies of Euclid was precisely to keep the concept of motion out of geometry, as a matter of principle, in accordance with certain philosophical considerations (see p. [188]). But then the *abstract concept of congruence* should be at the head – as in our second foundation – and then this fourth postulate would have to serve as the *basis for the theory of congruence*. The question arises here, to be sure, why analogous statements are not also made concerning the congruence of

segments. But we shall soon see what grave difficulties result from each of these points of view, in the further developments in Euclid.

Let me remark that neither of the two interpretations adequately explains why this theorem is found among the *postulates* whose general tendency is characterised above. This has called forth an interesting explanation from Zeuthen, which is not wholly convincing. He argues that the postulate would state that the prolongation of a segment beyond a point, which by postulate (b) is certainly possible, is *unique*. The details are to be found in Zeuthen's *Geschichte der Mathematik im Altertum und Mittelalter*.[88] Finally, there is always this loophole, the assumption that *the text here has been altered*. Indeed, this conclusion has been reached repeatedly and it cannot, in fact, be silenced.

I turn now to the *axioms*, of which there are again five in the Heiberg edition:

(a) *Things equal to the same thing are equal to each other; if* $a = b$, $b = c$, *then* $a = c$.
(b) *Equals added to equals give equals; if* $a = b$, $c = d$, *then* $a + c = b + d$.
(c) *If* $a = b$, $c = d$, *then* $a - c = b - d$.
(d) *Two coincident things are equal.*
(e) *The whole is greater than a part*; $a > a - b$.

[215] Four of the propositions just stated are logical in nature, and, as introduced here, they are obviously intended to state that the general relations which they express hold, in particular, also for all the geometric quantities, which occur (segments, angles, areas, etc.). The fourth statement, then, declares that the deciding criterion as to equality or inequality is, ultimately, *congruence* or *coincidence* – whereby, to be sure, it is again not clear whether or not the idea of motion is assumed.

Concerning the *difference between axioms and postulates*, Simon has advanced the idea that the former have to do with the *simplest facts of logic*, while the latter deal with those of *space intuition*. This would be very fitting and illuminating if it were only certain that the order in the Heiberg text corresponded to that in the original. In the various manuscripts, however, there are actually essential divergences, both as to order and as to content of the postulates and axioms, which by no means fit into this scheme; e.g., the *parallel postulate* is often entered as the eleventh axiom.

The Beginning of the First Book

Now we shall examine more closely the *beginnings of the Euclidean teaching structure of geometry,* which is built upon these definitions, postulates, and axioms, namely, the first four paragraphs, which immediately follow the axioms. In this we shall be able, at the same time, to make some interesting observations concerning Euclid's conception of the foundations, in particular his attitude toward the idea of motion.

[88] *Loc. cit.*, pp. 123–124.

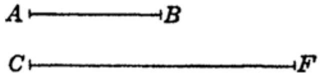

Figure 128

The purpose of the first three paragraphs is to solve the problem of *laying off a given segment AB upon another segment CF, beginning at C* [see Fig. 128]. Practically, anyone would, of course, do this by direct transference, using a compass or a strip of paper, i.e., by displacing a rigid body in the plane. Euclid does it otherwise with his theoretical method. In his postulates, he has assumed no construction, which corresponds to this free movement of the compass. His postulate (c) (see p. [213]) permits the drawing of a circle about a point only when a point of the periphery is already given. Now he may make use only of the possibilities afforded by the postulates, and he must therefore break up this apparently simple construction into a number of more complicated, but very clever, steps:

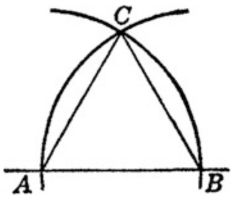

Figure 129

1. *Upon a given segment AB to erect an equilateral triangle* [see Fig. 129]. Postulate (c) permits us to draw a circle about A with radius AB, and one about B with radius BA. That these circles will have a point of intersection C is, as mentioned above, assumed without any explanation. Then follows a rigorous formal logical **[216]** proof, with use of the appropriate axioms, that ABC is actually equilateral.

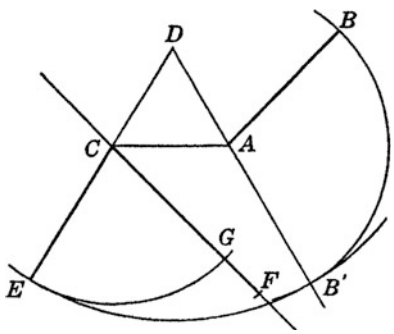

Figure 130

2. *To lay off from a given point C a segment equal to a given segment AB* (see Fig. 130). By (1), erect upon AC an equilateral triangle ACD. Prolong DA beyond

A (Postulate b), and strike a circle about A with radius AB (Postulate c), so as to get the intersection B' with DA. (The reason for the existence of this intersection is, to be sure, again not explained.) Now draw a circle about D, with a radius DB', and obtain its intersection E with the extension of DC; then $CE = AB$. The proof, which is obvious, is then given in detail.

3. *Given two segments AB, CF, such that $CF > AB$; to lay off from C upon CF a segment equal to AB.* By (2), draw from C any segment $CE = AB$ and describe about C a circle, with a radius CE, meeting CF in G; then CG is the desired segment.

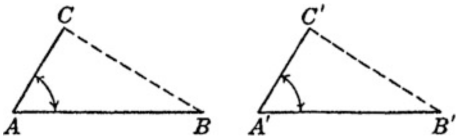

Figure 131

With this, the given problem is solved. Euclid now states, as No. 4, the *first congruence theorem*: If two triangles ABC and $A'B'C$ have, in each, two sides and their included angle respectively equal ($AB = A'B'$, $AC = A'C'$, $A = A'$), the triangles are equal in all their parts. In proving this theorem, Euclid is guilty, in view of the preceding construction, of that noteworthy *inconsistency*, which supplies the reason for my reproducing this entire proof. He thinks of the triangle $A'B'C'$ laid upon ABC so that the sides $A'B'$ and $A'C'$ fall respectively upon AB and AC, and angle A' upon A. Now we have learned, indeed, in what precedes, how to lay off a *segment* upon another, but not a word has been said as to the laying off of an *angle*, and still less about what would happen, in this process of transfer, *to the third side $B'C'$*, not even whether or not it would, indeed, remain a *straight line*. Intuitively this is, of course, quite clear; but Euclid's entire purpose is the *logical* completeness of the deduction. Nevertheless he concludes here, without further explanation, that $B'C'$ must also go over into a straight line, which must then, of course, coincide with BC. However, this is nothing else than the assumption of motions, which
[217] do not change the form and the measurements of the geometric figures – just as we explicitly did do in our first foundation of geometry. If this is done, it is then obvious, of course, that the first congruence theorem can be proved (see p. [189]).

Thus this proof of Euclid's would seem to show that he was a supporter of the idea of motion. The question then remains as to why nothing is said about it in the foundations. Above all, his skilful proof of Exercises 2 and 3 would then be without purpose, since that proof could be given in a word by use of the concept of motion. On the other hand, however, if we look upon No. 4 as a later interpolation, the question is still open as to what Euclid's attitude may have been toward the first congruence theorem. Hence there remains an *essential gap* in his development. Without the concept of motion, it is impossible to prove this theorem and we must place it, as we did in our second foundation, among the axioms (p. [189]). We can only say, in concluding this discussion *that so many essential difficulties present themselves, precisely in the first theorems of the first book of the Elements*,

that there can be no question of having attained that ideal, such as that mentioned above.

The Lack of "Betweenness" Axioms in the Elements; the Possibility of the So-Called Geometrical Sophisms

But all these gaps and obscurities do not weigh so heavily as *another objection* which must be made to Euclid's presentation of the foundations if one measures him by his own ideal and at the same time considers our present knowledge. If we resort to the familiar language of analysis, Euclid, with his geometric quantities (segment, angle, surface, etc.), *never uses a sign – he* treats all of these as absolute quantities. He carries on, in a sense, an analytic geometry in which the coordinates and other quantities appear only with their absolute values. The result of this is that he cannot obtain theorems that have general validity, but must always drag along different cases according as, in a concrete instance, the parts lie thus or other. To mention a simple example, the so-called extended Pythagorean theorem, expressed in the modern formula $c^2 = a^2 + b^2 - 2ab\cos\gamma$, holds generally for triangles with acute or obtuse angles (see Fig. 132) since $\cos\gamma$ takes on both positive and negative values. But Euclid knows only the absolute value $|\cos\gamma|$ and he must therefore distinguish the two cases in two different formulas:

$$c^2 = a^2 + b^2 - 2ab|\cos\gamma| \quad \text{and} \quad c^2 = a^2 + b^2 + 2ab|\cos\gamma|\,;$$

of course these case distinctions become more complicated and less perspicuous the farther one goes.

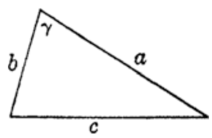

Figure 132

This lack of which we are talking can of course be formulated for pure geometry. **[218]**
A difference in sign in the analytic presentation corresponds, in pure geometry, to a difference in order, of the type as to whether a point C lies between the points A and B, or outside the segment AB. It is possible to realise a completely logical construction of geometry, only if we expressly formulate the fundamental facts in this relation of position, the so-called "*axioms of betweenness*," as we did, with emphasis, in our first, as well as in our second, foundation of geometry. If we omit this, as Euclid does, we cannot reach the ideal of a pure logical control of geometry. We must continually recur to the figure and we must discuss these relations of position. *Our objection, then, against Euclid is, in brief, that he has no axioms of betweenness.*

This view that one must actually formulate certain assumptions concerning the concept "between," in other words, that we must endow the elementary quantities with signs, according to certain conventions, is relatively new. At the beginning of this lecture course (p. [17]), when we discussed this topic, I reported that the first consistent use of the rules of sign is to be found in Möbius' *barycentric calculus*, in 1827. In this connection there is an interesting letter from Gauß to Wolfgang Bolyai, dated March 6, 1832, but first published in 1900 in volume 8 of Gauß' works,[89] in which we find: "For complete achievement, we must first base such words as 'between' upon clear concepts, a thing which is quite feasible but which I have nowhere seen done."

The first careful geometric formulation of these "axioms of betweenness" was given by Moritz Pasch in 1882 in his *Vorlesungen über neuere Geometrie*.[90] Here there appeared for the first time the characteristic theorem, which we used, by the way, in our first foundation of geometry (p. [178]): *If a straight line meets one side of a triangle*, *it also meets one of the other two sides*. (See Fig. 133.)

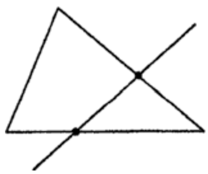

Figure 133

The significance of these axioms of betweenness must not be underestimated. They are just as important as any of the other axioms, if we wish to develop geometry as a really logical science, which, *after* the axioms are selected, no longer needs to have recourse to intuition and to figures for the deduction of its conclusions. Such recourse is, however, stimulating, and will of course always remain a necessary aid in research. Euclid, who did not have these axioms, always had to consider different cases with the aid of figures. Since he placed so little importance upon correct
[219] geometric drawing, there is real danger that a student of Euclid may, because of a falsely drawn figure, come to *a false* conclusion. It is in this way that the numerous so-called *geometric sophisms* arise. These are formally correct proofs of false theorems, which rest on figures, which are wrongly drawn, i.e., which contradict the axioms of betweenness. As an example, I shall give one such sophism, which is certainly known to some of you, the "*proof*" *that every triangle is isosceles*.

Draw the bisector of the angle A, and the perpendicular to the side BC at its middle point D. If these two lines were parallel, the angle bisector would be also the altitude, and the triangle would obviously be isosceles. We assume then that these two lines meet, and we distinguish two cases, according as the meeting point

[89] Page 222.

[90] Leipzig, 1882 (2nd edition, 1912).

O lies inside or outside the triangle. In each case, draw OE and OF perpendicular to AC and AB, respectively, and join O to B and to C.

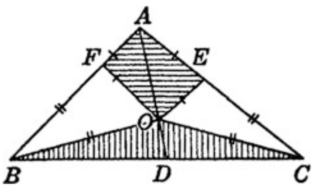

Figure 134

In the first case (see Fig. 134), the horizontally hatched triangles AOE and AOF are congruent, because the side AO is common, and the angles at A are equal, as are also the right angles. Hence $AF = AE$. Similarly the vertically hatched triangles OCD and OBD are congruent, since OD is common, $DB = DC$, and the right angles are equal, so that $OB = OC$. Now, because, from the first congruence, $OE = OF$, we can infer the congruence of the unhatched triangles OEC and OFB. Hence we have $FB = EC$, and, adding this to the former equation, we get actually $AB = AC$.

In the second case, where O lies outside (see Fig. 135) we show, in the same way, the congruence of the three pairs of corresponding triangles, and we find $AF = AE$, $FB = EC$. By subtraction it follows, again, that $AB = AC$, as the figure shows. Hence it is proved that in every case the triangle is isosceles.

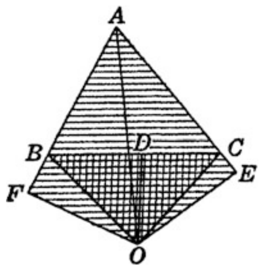

Figure 135

The only thing in this proof that is false is the figure. In the first place, *O can never fall inside the triangle*; and, in the second place; the positions can never be as they are drawn in Fig. 135. Of the two feet E and F, of the dropped perpendiculars, *one must lie inside*, *the other outside* the side on which it lies, as shown in Fig. 136. **[220]**
Actually, then, we have

$$AB = AF - BF, \quad AC = AE + CE = AF + BF,$$

and we can by no means infer the equality of the two sides.

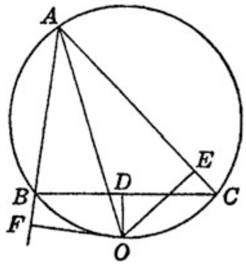

Figure 136

This clears up this sophism completely, and we can dispose in a similar way of the many other known sham proofs. The argument is always based upon inaccurate figures, with perverted order of points and lines.

The "Archimedean" Axiom in the Elements; Excursus About the "Horn-shaped" Angles as an Example as a System of Quantities Excluded by this Axiom

Having criticised some of the essential shortcomings in Euclid's exposition I should like to point out one of its greatest *refinements*, one which his enthusiastic supporters usually overlook, as they do his errors. I remarked earlier that in the fifth book the *ratio* (logos) of any two geometric quantities a and b is so considered that it gives the equivalent of the general concept of numbers. Now Euclid stipulates here expressly that he will consider the ratio of two geometric quantities of the same sort *a and b only under a certain condition*: *if*, *namely*, *two integers m and n can be found such that* $ma > b$ *and* $a < nb$. His words are: *Quantities have a ratio if their multiples can exceed one another*. This requirement is called nowadays *Archimedes' axiom*, a name which is thoroughly at variance with history, since Euclid had it before Archimedes, and it is probable that Eudoxus knew it. Today the designation *axiom of Eudoxus* is gaining currency.

This archimedean axiom plays a great role as one of the most important continuity postulates in modern investigations in the foundations of geometry, as well as in the foundations of arithmetic. We have accordingly mentioned it repeatedly in our own expositions. You will notice, in particular, that the postulate which we used in our first foundation of geometry, whereby the points arising from A, through iteration of a translation, ultimately include every point of a half straight line (p. [175]), is identical in substance with the archimedean axiom. But we also discussed this axiom in detail in the first part of this present work.[91] We then called a quantity a which, after multiplication by any finite number n, remained always smaller than b, *actually infinitely small with respect to b*, or conversely, *b actually infinitely large*

[91] See Vol. I, p. [235].

with respect to a. Thus what Euclid does, by his prescription, is to exclude systems [221]
of geometric quantities, which contain actually infinitely small or infinitely large elements. In fact, it is necessary to exclude such systems, if we wish to develop the doctrine of proportion, which, as we have emphasised, is nothing else than another form of the modern theory of irrational number. Thus Euclid (or, indeed, Eudoxus before him) does here – and that is the remarkable part of it – fundamentally exactly what one does in the modern investigations of the number concept , and he does it with exactly the same tools.

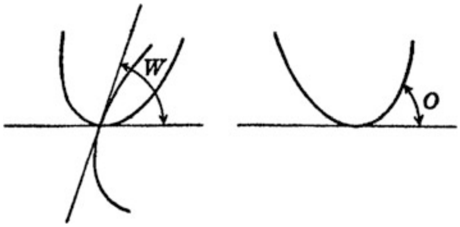

Figure 137

We shall appreciate best the significance of the axiom under discussion if we examine a *concrete system of geometric quantities*, *which does not satisfy it*, and which is also particularly interesting because it was already known and much discussed in ancient and in medieval times. I refer to the so-called *horn-shaped angles*, that is, *angles between curves*, thought of in a certain general way. When we speak today of angles, we think always of angles between straight lines; and by the angle between two curves, in particular, we understand the angle between their *tangents* (Fig. 137). The angle between a curve, say a circle, and its own tangent is then always zero. In this way, all angles form an ordinary "archimedean" system of quantities, to which we can apply the Euclidean theory of ratio, which, in other words, is measured in terms of simple real numbers.

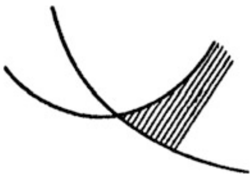

Figure 138

In contrast to this, we understand by the *horn-shaped angle between two curves* (see Fig. 138) the *portion of the plane enclosed* by the curves themselves, in the neighbourhood of their intersection (or point of contact), and we shall now see how this definition gives rise to the concept of a *non-archimedean quantity*, i.e., *to a concept, which does not satisfy that axiom.* We shall confine ourselves, here, to angles where *one of whose arms is a fixed straight line* (the x-axis), whose vertex

is the origin O, and whose other arm is a circle (in case of need also a straight line), which cuts or touches the x-axis in O (Fig. 139). It will then be natural to call that one of two horn-shaped angles the smaller *whose free arm ultimately remains*
[222] *below the free arm of the other*, *when we approach* O, i.e., the one, which ultimately bounds the narrower portion of the plane. The angle of a *tangent* circle will thus always be smaller than that of an *intersecting* circle or of a straight line. Of two tangent circles, the one with the *larger* radius will make the *smaller* angle, since it passes below the other. It is clear that these agreements determine, for any two of our horn-shaped angles, which of them is the smaller and which the larger, so that the totality of horn-shaped angles is *simply ordered*, as one says today in set theory, precisely as is the case with the totality of ordinary real numbers.

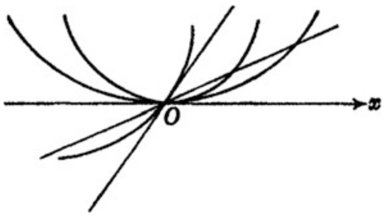

Figure 139

In order to appreciate the difference between these two sets, we must agree upon something more precise concerning the *measuring of horn-shaped angles*. Let us, first of all, measure the angle of a straight line through O in *ordinary angle units*. Then every angle a, made by a circle tangent to the x-axis, will be smaller, by definition, than any angle bounded by two straight lines, however small it may be, provided only that it is but different from zero. Such a situation is impossible, however, in the ordinary number continuum, for a number a different from zero, and it characterises our a as "actually infinitely small."

In order to follow this in connection with the archimedean axiom, we must define, for these curvilinear angles, *multiplication by an integer*. If we have a circle of radius R tangent at 0, then it seems natural to ascribe to the tangent circle of radius R/n the n-fold angle. This actually accords with the preceding definition, insofar as the angles of tangent circles with radii R, $R/2$, $R/3$, ..., get larger and larger. Thus multiplication of the angle a of a tangent circle by an integer always yields another angle of a *tangent* circle, and every multiple na is necessarily smaller, by our definition, than, say, the angle b of a fixed intersecting straight line (see Fig. 140), however large we take n. Thus the *axiom of Archimedes is not satisfied*; and the angles of the tangent circles must be looked upon, accordingly, as actually infinitely small with respect to the angle of an intersecting straight line. As to *general addi-*
[223] *tion* of two such angles, that will be done, in view of the definition already set up for multiplication by integers, by adding the reciprocal values of the radii, which will serve, after all, as the measures of the actually infinitely small angles.

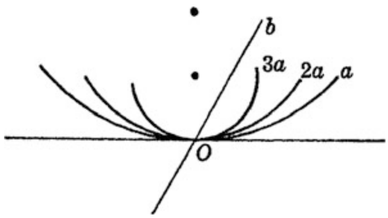

Figure 140

If we have now an *arbitrary circle through O* (see Fig. 141), we can look upon its angle as the sum of the angle of its tangent with the x-axis (measured in the ordinary sense), and of its own actually infinitely small angle with that tangent, in the sense just defined. If we then apply addition and multiplication to these separate summands, we shall have set up a complete method for operating with horn-shaped angles. But in this field the axiom of Archimedes does not hold, and one may not, therefore, employ in it "logoi," or ordinary real numbers. Presumably, this was known to Euclid (and Eudoxus), and he consciously excluded such systems of quantities by means of his axiom.

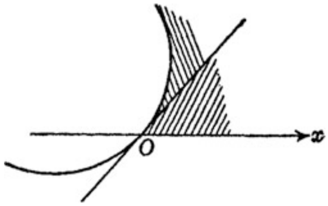

Figure 141

With modern methods we can *extend* the field of these horn-shaped angles, whereby the definitions become both broader and simpler – if we consider *all of the analytic curves through 0.* Any such curve will be given by a power series $y_1 = \alpha_1 x + \beta_1 x^2 + \gamma_1 x^3 + \ldots$, $y_2 = \alpha_2 x + \beta_2 x^2 + \gamma_2 x^3 + \ldots$ We shall say that the angle of the curve 1 with the x-axis is greater or less than that of 2 according as $\alpha_1 > \alpha_2$ or $\alpha_1 < \alpha_2$ if, however, $\alpha_1 = \alpha_2$, then relative size depends upon the inequalities $\beta_1 \gtrless \beta_2$; if $\beta_1 = \beta_2$, then the decision rests upon the inequalities $\gamma_1 \gtrless \gamma_2$, etc. It is clear that, in this way, *the angles of all analytic curves are brought into a definite simply ordered series*, in which circles are included in the order defined for them above.

To get the n-fold of the angle of curve 1 with the x-axis, we can simply take the angle of the curve $n \cdot y_1 = n\alpha_1 x + n\beta_2 x^2 + \ldots$, obtained by multiplying the power series by n. Before, we had to employ a more complicated operation, in order not to get outside the field of circles; namely, we replaced the circle of radius R, whose

series expansion is

$$y = \frac{x^2}{2R} + \frac{x^4}{8R^3} + \dots$$

by one of radius R/n:

$$y = n\frac{x^2}{2R} + n^3\frac{x^4}{8R^3} + \dots,$$

which agrees only to the first term with n times the first expansion. However, with
[224] this new and simpler definition we also have again a non-archimedean system of quantities. A curve whose series expansion begins with x^2 (while $\alpha_2 = 0$) will, after multiplication by arbitrarily large n, still make a smaller angle than a curve with non-vanishing α_1. We have, in essence, only repeated here, in more intuitive form, what we did in volume 1.[92] In the power series $y = \alpha x + \beta x^2 + \gamma x^3 + \dots$, the successive powers x, x^2, x^3, ..., simply play, in this interpretation, the role of actually infinitely small quantities of different, ever-increasing order.

It is interesting that we can condense this succession of horn-shaped angles still more by adding *certain non-analytic curves*. However, in order to permit of comparison as to size, they must not oscillate infinitely often, or, more precisely, they may not cut an analytic curve infinitely many times. It will suffice if I give one example, the curve $y = e^{-1/x^2}$. This curve has the property that all its derivatives vanish at $x = 0$. Hence it does not permit there an expansion in power series. It is clear, therefore, that it ultimately passes below *every* analytic curve. Notwithstanding the fact that we had before a dense succession of horn-shaped angles, we have now a *new horn-shaped angle*, *which*, *together with its finite multiples*, *is smaller than any angle made with the x-axis by any analytic curve*.

With this we shall conclude these discussions and our entire study of Euclid. In closing, I shall summarise, in a few sentences, *the judgment concerning Euclid's Elements*, *which we have reached in all these analyses*.

1. The *great historical significance* of Euclid's *Elements* consists in the fact that through them there was passed on to later times the *ideal of a consistent logical exposition of geometry*.

2. *As to its execution*, *much of it is very nicely done*; *much of the remainder*, *however*, *remains decidedly below our present scientific standpoint*.

3. Numerous *details of an important nature*, especially at the beginning of the first book, remain *doubtful*, because of uncertainties in the text.

4. The entire exposition seems often *unnecessarily cumbersome*, because Euclid had *no arithmetic* ready at hand.

5. *One-sided emphasis on the logical processes* renders difficult both the understanding of the work as a whole, and its essential connections.

I should like to characterise farther our own attitude toward the foundation of
[225] geometry, by recalling two conceptions, which have already been noticed at different points.

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92 Part I, p. [218] et seq., where the magnitudes of different orders were called η, ζ, ...

One of these has to do with the fact that we were able to give the foundations of geometry according to entirely different plans. We gave careful attention to two of these. The one method started with the notion of a group of motions, in particular the group of translations. The other began with the axioms of congruence and pushed parallelism to a much later place. This juxtaposition gives prominence to the freedom, which we have in the axiomatic foundation of geometry. And I should like especially to emphasise again this fact, in the face of intolerant utterances which one often hears, and which are aimed at championing this or that pet concept of the author, as *absolutely* the simplest and, in fact, the *only* suitable one to use in the foundations of geometry. As a matter of fact, the source of all fundamental geometric concepts and axioms is our naive geometric intuition. From it we choose the data, which, in appropriate idealisation, we lay at the base of the logical treatment. As to which choice should be made, however, there can be no *absolute* judgment. The freedom, which exists here is subject to only one restriction, namely, the requirement that the system of axioms shall fulfil its purpose of guaranteeing a consistent construction of geometry.

Another observation concerns our attitude to analytic geometry, and our criticism of certain traditions, from Euclid on, which have long since ceased to conform to the position of mathematical science, and which should, on that account, be given up in school mathematics. In Euclid, geometry, by reason of its axioms, is the rigorous foundation of general arithmetic, including also the arithmetic of irrational numbers. Arithmetic remained in this position of bondage to geometry well on into the nineteenth century, but since then there has been a change. Today arithmetic has reached a dominating function as the proper fundamental discipline. This is a fact, which ought to be reckoned with in the foundation of scientific geometry, i.e., geometry should relate to the results of arithmetic. The attitude to analytic geometry which we took in our foundation, and the fact that we have systematically made use of the resources of analysis in the treatment of geometry, merit approval in this sense.

With this we bring to a close our analyses of the theories of pure geometry, with the hope that they have given you the desired survey of the whole field, insofar as it has any relation to the needs of the schools. And now, at the end, we want to treat comprehensively, as I already announced, somewhat of the teaching of geometry.

Final Chapter: Observations About the Teaching of Geometry [226]

Importance of the Historical Background

Here, the presentation will naturally get much more historical in character, even more so than in the corresponding discussions in the first volume; since geometry can – thanks to its venerable age as a science – also look back on such a long tradition as a subject of teaching, unlike the earlier disciplines discussed. While this tradition offers, on the one hand, an advantage, it brings, on the other hand, serious dangers in other respects. In fact, *geometry teaching suffers today almost because of the burden of tradition*, since many no longer viable components have now taken root so firmly that they are difficult to eliminate and even considerably impede the introduction of new sound topics.

To understand the current state of geometry teaching,[93] we must go back to the time of the reawakening of scientific activity, to the *Renaissance*, taken in the broadest sense (from 1200 on). In those times, it was self-evident that one was inspired by the Ancients and studied Euclid's *Elements* in particular as an introduction to geometry. Then one studied other components of the geometry of the Ancients that were preserved, that is, primarily the calculation of π by Archimedes, the conic sections by Apollonius, and, finally, the interest in constructions with ruler and compass, which originated in the Platonic school. Such geometry teaching is naturally extremely one-sided; not only the concern for applications, but also the formation of space intuition were completely marginalised, the emphasis was exclusively on the abstract logical side of geometric deduction. Now, what is strange in all this is that not only the researcher, the scholar studied geometry in this manner, but that the view became established that Euclid's Elements were a textbook suitable for the first grades of teaching! While this confusion might have been obvious in this [227]

[93] Additions to this presentation, already written in 1908, are given in appendix 2, at the end of the present volume.

period, since one had nothing else than Euclid, this surely did not correspond to Euclid's own opinion: since the Elements arose – one cannot emphasise this often enough – from university lecture courses[94] and are far from a textbook for ten-year-old boys. And yet, this misconception has essentially persisted up to the present time, as we see ever more frequently.

Contrasting Modern Requirements

Let us first ask, what requirements should be made today of a sound geometrical education. Everyone will surely admit for this that:

1. *The psychological aspects* must substantially prevail. Teaching cannot only depend on the subject matter, but it depends above all on the *subject* that you have to teach: one will present the same topic to a six-year-old boy differently than to a ten-year-old boy – and this, in turn, differently to a mature man. Applied in particular to geometry, this means that in schools you will always have to connect teaching at first with vivid concrete intuition and then only gradually bring logic elements to the fore; in general, the genetic method alone will provide a legitimate means slowly to develop a full understanding of concepts.

2. As regards the selection of teaching topics, we will choose, from the entire field of pure and applied geometry, such pieces that seem to correspond to the aims of geometry in the frame of education as a whole without being influenced by historical contingencies. It is not unnecessary, ever to ask for general demands of this kind; since – although everybody is inclined to admit them theoretically, they are often not sufficiently followed in practice.

3. With regard to the general educational aims, I cannot go into the finer nuances between the different types of schools. Let it suffice to point out that it greatly depends on the particular cultural orientation of the respective period; and we will certainly not plead for a flat utilitarianism, when we indicate as the aim of the modern school, to make wide classes morally and intellectually capable of participating in the cultural work of the present times, which is substantially oriented towards practical activity. Therefore, especially for mathematics teaching, an ever greater concern for the natural sciences and technology proves to be necessary.

4. Clearly, I cannot offer a definite selection of teaching topics; only the teacher as expert in the practice decides it; he, who himself possesses a rich experience of teaching. The present lecture course is intended, as I have often already emphasised, to provide a basis for such a selection insofar as it gives you an overview of the
[228] whole of pure geometry, and of the material at hand, which will enable you to make a proper expert judgment about this question.

[94] [Translator's note: Since no universities existed in Greek and Hellenistic times, Klein wants to say here that such studies were pursued by adults.]

5. I should like to emphasise only one useful methodological view-point, namely the already repeatedly mentioned tendency for a fusion of planimetric and stereometric teaching that seeks to prevent a one-sided teaching of planimetry while neglecting three-dimensional space intuition. And in the same sense one has still to require a fusion of arithmetic and geometry. I do not mean that these areas should be completely merged, but they should not be so sharply divided, as still seems to happen nowadays in schools. My entire lecture course shows how I wish this to be understood.

Criticism of the Traditional Teaching Mode

If you measure present actual school practice against these ideas and requirements, it will prove in many respects to be by no means satisfying. Of course, it is difficult to make a general judgment because even within the same country the same practice will not dominate in all schools, it might even differ from teacher to teacher; but I think that I can prove a few patterns to be correct in a large measure, even when one can indicate for each of my criticisms numerous cases where it is not valid.

1. Above all, I believe that the fusion of different disciplines nowadays is too little realised in the classroom; I want to prove this by giving some details that you might still have in living memory:

a) Projecting and drawing of spatial figures, which is certainly something very important, is not given this role in today's geometric teaching. It might probably be added to the syllabus, as an external element, but not united conceptually with it. A related concern is that what one calls the "spirit of modern geometry" is also not given the role it merits in the classroom: I mean the *idea of mobility of each figure*, by which in any case, beyond any specific one, the general nature of geometric structures becomes understandable. Admittedly, one has some items of "modern geometry" introduced into the syllabus, such as the concept of the harmonic points and of the transversals; but what allows us to catch the essence of its method at a glance is typically dissolved into numerous case distinctions, in the rigid Euclidean style.

b) Geometry and arithmetic are usually kept at school unnaturally separated; a revealing example is the teaching of proportions already mentioned above (p. [208]), taught at first arithmetically and then – often without connecting to the former teaching – in geometric form.

c) Analytical geometry with the principle that a function $y = f(x)$ is a curve is **[229]**
certainly accessible to the perception of boys already at an early stage, and could and should from then on penetrate all the geometric teaching. Instead, it is superimposed as a new separate building on the finished geometry, and if the conic sections have been treated once "synthetically" (in the meaning of the Ancients!) it will be shown then how, by means of a "new discipline", of analytic geometry, one can treat the topic a lot easier. The deeper conception of modern historical research that

the ideas of analytic geometry were basically already present in Apollonius, do not become implemented there though.[95]

2. I would now like to take a look at the consequences for science of this insistence of teaching in the historically given separation of the individual areas. Evidently, elementary geometry even in its historical limitation, deplored by me, offers numerous occasions for scientific problems. As for literature, I would refer only to Max Simon's talk "On the development of elementary geometry in the 19th century",[96] on the other hand, besides my booklet "Lectures on selected questions of elementary geometry",[97] I should mention the interesting collection by Federico Enriques: "Questioni riguardanti la geometria elementare", which is also published in German translation, entitled "Fragen der Elementargeometrie" in 2 volumes;[98] finally, the "Theory of geometric constructions" by August Adler should be emphasised.[99]

Unfortunately, I cannot go into the positive side of the interesting problems arising here, rather I have to confine myself to emphasising some bad grievances that have emerged as a result of the isolated position of elementary geometry, far away from the general development of mathematics. Some particular areas have been detailed and extended too much and also introduced into the classroom, which, from a higher point of view have little or very little interest.

a) In this regard, I have at first to mention the discipline known in school mathematics as *algebraic geometry*: calculating first parts of the triangle or any other figures and then, what is taught separately, constructing with them. You have a means for rating the value of these areas, when you ask if you have ever used, **[230]** or could have used, them in higher education. Certainly not; this is just a minor side branch that has been artificially maintained only for its own sake and never came into lively interaction with other branches of science.

b) Also famous is the area of triangle constructions. That one ever constructs figures is very nice and helpful and I recommend certainly always the use of graphical procedures in all fields. You have just here in Göttingen thanks to Runge's lecture courses on graphical methods the best opportunity to get acquainted with the numerous highly ingenious methods developed in recent times.[100] But in school one is not concerned with these generally important and interesting issues; rather, one is restricted primarily to the construction of triangles and in particular to tasks that are solvable with ruler and compass. As is known, one obtains a great variety of such partially quite difficult tasks if one chooses the three given parts of the triangle

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[95] [Translator's note: These results of historiographical research, modern in Klein's days, have been revised since; see Michael Fried & Sabetai Unguru, *Apollonius of Perga's Conica: text, context, subtext*. Leiden: Brill, 2001]

[96] *Jahresbericht der deutschen Mathematikervereinigung*. Ergänzungsband I. Leipzig 1906.

[97] Ausgearbeitet von F. *Tägert*. Leipzig 1895.

[98] Bologna 1900 [3rd edition 1924].

[99] Sammlung Schubert 52. Leipzig 1906.

[100] See, for instance: *Carl Runge,* Graphische Methoden. 2. Aufl. Leipzig 1919 (Sammlung mathematisch-physikalischer Lehrbücher 18) and *Horst von Sanden*, Praktische Analysis. Leipzig 1914 (Handbuch der angewandten Mathematik 1).

in the most diverse and – as has been said in – “the most inappropriate possible manner”. To be sure, one quite often does not place real emphasis on the actual carrying out of the resulting constructions; and, in fact, they are usually also much too complicated in practice, due to the artificial restriction of the means allowed. Certainly, there are theoretically very interesting deep questions which can be linked with such constructions, as they are treated as in the work of Enriques mentioned, or as they have been discussed with respect to some examples in the first volume of this work:[101] I mean the algebraic proofs of impossibility that show why, in certain constructions (for instance the construction of the regular heptagon or the trisection of any angle) compass and ruler are explicitly *no* longer sufficient. However, school teaching does not touch upon this even allusively, so that, unfortunately, again and again many people obtain the firm belief that each geometric task has to be and can be performed with ruler and compass. This probably constitutes also the reason why the phenomenon of the enormous crowds of those pretending to have squared the circle and trisected the angle, of which I spoke to you in the previous semester, will never die off.

c) Finally, I have to mention the so-called triangle geometry, which is the doctrine of the “remarkable” points and straight lines of the triangle, which has been especially developed within school mathematics as an independent discipline; here you will agree with me that this area has become marginal in higher education in **[231]** the same measure as it used to enter the foreground in the classroom. I have already explained the corner of projective geometry in which this triangle geometry has to be ranged (see pp. [170] sqq.): it is the theory of invariants of that plane figure, formed by three arbitrary points and the two imaginary circular points of its plane, and so something actually quite particular.

If we want be more specific regarding the present state of geometry teaching, beyond this general criticism, we have to study separately the development in individual countries, since it is, of course, realised quite differently everywhere; here, we can only analyse the most important civilised countries, such as England, France, Italy and Germany.

[101] See Volume I, pp. [54] sqq. (heptagon), pp. [122] sqq. (trisection).

I. The Teaching in England

The Traditional Type of Teaching and the Exams

England is the country, which was for the longest time under the spell of the medieval tradition of Euclid and that continues to have an effect there, at least partly, even today. This situation is due to the organisation of the English examinations. The beautiful principle that one should learn independently of exams is unfortunately, like so many other beautiful principles, followed nowhere. In England, moreover the strange system of strictly centralised exams dominates while otherwise there is a completely independent, private organisation of the individual schools. It is just the reverse of our system: in Germany, the students are examined by the teachers at each school who know them well, and thus their individuality can largely be considered. On the other hand, we have uniform curricula that prescribe certain general guidelines regarding the teaching issues and the methods for all schools. In contrast, in England, the individual schools and private institutions have almost complete freedom of movement and are of the most heterogeneous nature regarding their entire organisation. However, they are not entitled to *examine* their students themselves. Instead, there is the principle that the examiner does not know the student – he will not even see him; instead, he will only judge quite schematically his written performance – its result will decide exclusively the result of the exam. London, Cambridge and Oxford are the location of the big exam commissions, which examine entrants from the whole country. London, for instance, as reported to me by one of the principal examiners, examines 24,000 students annually, and they all receive the same tasks, the same questions. The examiner has 30 assistants for checking these exam papers, each of whom has therefore to correct the same work some 800 times. Of course, nobody would accept such a work if it were not paid very well.

In mathematics education such a peculiar method is only possible if there is **[232]** a "standard work" that any examinee knows and on which the examiner's questions can be based; as such a normal book functioning in England of old, as far as geometry is concerned, was the Elements of Euclid. It is understandable that in such a system that remained essentially unchanged, one and the same work and thus also one and the same teaching method remained for such a long time also essentially unchanged, and that in such a system realising a reform will meet the greatest dif-

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_14

ficulties. The exams authorities cannot by themselves reform the organisation of teaching throughout the country,[102] since they have no official influence at all on it, and on the other hand, it is difficult for them, given that an enormous number of exams have to be handled, to consider special situations at an individual school which might want to make independent experiments with new teaching methods.

Now let's look at one such English school-Euclid. I am describing here to you the edition by Robert Potts,[103] which in recent decades became particularly widely used. It contains only – that is characteristic – the books 1 to 6 (planimetry) and 11, 12 (beginnings of solid geometry and the exhaustion method), and, in fact, the entire text is given in literal translation. Explanatory and historical notes are added, as well as exercises. From the original elements, therefore, the following are missing: the arithmetic books 7 to 9, the classification of irrationalities in 10 and the regular solids in 13. This curricular substance is traditionally learned in the English schools more or less by heart, so that the student will have it at his disposal in the examination. Perry once made the amusing remark to characterise this method: "How healthy the English nature must be that it has endured through the centuries with such an unsuitable method of education." Admittedly, the need had been felt to take account of modern research, going far beyond Euclid. This one did but by pressing the new by force into the rigid Euclidean form – thus, of course, losing a large part of the modern spirit. As an example of the resulting so-called "sequels to Euclid", I mention here the book by John Casey,[104] which deals with the elements of projective geometry in this manner.

The Association for the Improvement of Geometrical Teaching

Of course, a reaction against this rigid system was inevitable; it drew on a suggestion made by the great English mathematician Sylvester in 1869 and led in 1874 to the founding of an *Association for the Improvement of Geometrical Teaching*.
[233] This society worked for long until they finally achieved the issue of a new standard book, the *Elements of geometry*.[105] This is essentially only a slightly adjusted and smoothed revision of the first 6 books of the Euclidean elements. Thus, for instance, the roughnesses at the beginning of the first book, of which we complained, are eliminated and the concept of motion is consistently used as fundamental. In general, however, Euclid's order and limitation of subjects is maintained again – caused by regard to the examinations. So it's just a petty tame reform that is being attempted here, but it met sharp opposition by the adherents of the old Euclidean

[102] [Translator's note: Before the book by Georg Wolff (see vol. I, apendix I) was published, Klein could not be aware exactly of all the elements functioning: There was no over-riding Exams authority – there were different boards and they reacted to calls for change in different ways.]
[103] Euclid's elements of geometry. London 1869.
[104] A sequel to the first 6 books of the elements of Euclid, containing an easy introduction to modern geometry, with numerous exercises. Dublin 1900. [Translator's note: The mathematician Dodgson is better known as the author of *Alice's Adventures in Wonderland*' which he wrote under the pseudonym of Lewis Carrol.
[105] Part 1. & 2. London 1884. 1888.

system. As evidence, I show you a quite amusing book written by Charles Lutwidge Dodgson: "Euclid and his modern rivals".[106] Here the author goes quickly to put the Association into court, in the literal sense; for he makes no less than the Hell Judge Minos sit before Euclid and his modern rivals, namely the authors of more recent textbooks, especially Legendre, who all have to defend their books. But Euclid alone does well here, while the others, especially the improvers of Euclid of the Association, are soon dismissed together with their arguments.

It is impossible here to discuss details and I should only like to refer to a matter of general importance, which has validity also for the literature of other countries. Many people who write about educational issues, know almost exclusively the school literature of their *own* country and have no idea either of parallel efforts in other countries, or of the progress of pure science in the relevant areas, that is, in this case, the foundations of geometry. You can see this well with Dodgson, in whose book, with the exception of Legendre (who occupies a special position) only English school writers are named and where there is no consideration of ongoing scientific research into the foundations. This observation can often be made: comparative investigations of teaching within the various nations, as we do here, are still far from being sufficiently widespread.

Perry and His Tendencies

A much greater effect than the work of the Association has resulted from another action for reform – a reform of, one might even say, a revolutionary character which is associated with the name of Perry. John Perry was an engineer and taught at one of the largest technical institutes of London; he initiated a powerful movement that in the strongest terms opposed the unilateral logical training obtained by studying Euclid and wanted to replace it by a teaching entirely based on intuition. This reformed teaching should result mainly in the complete mastery of how to *use* mathematics. Perry is best known as an author of textbooks, which aim to introduce engineers to **[234]** master the infinitesimal calculus in a practical mode. I mention particularly "Calculus for engineers",[107] which has been translated into German by Robert Fricke and F. Süchting as "Höhere Analysis für Ingenieure"[108]. In addition, as a characteristic of Perry's tendencies I mention the booklet "Practical mathematics"[109] which grew out of lectures given to classes for workers and tried in a very swift and thrilling manner to make the concepts of the coordinate system, of function etc. accessible to a larger public, by means of constant references to practical examples.

None of this is actually geometry, but by Perry's action one has tried to reform the teaching in this area, too, by introducing the so-called *laboratory method*. One begins there by teaching the students concepts in their practical application; stu-

[106] Second edition, London 1885.
[107] London, 3rd edition, 1899.
[108] Leipzig 1902. [4th edition 1923]
[109] London 1899.

dents have to draw curves on graph paper and to measure them, one exercises the use of the planimeter, etc. There is no emphasis on logical deductions and proofs or at least these approaches are greatly reduced. Only practical skills count. We have there what is actually the greatest possible contrast to Euclid's method. These approaches are tellingly expressed in the textbook by Harrison: "Practical planning and solid geometry for elementary students",[110] which in fact starts with indicating all one needs for drawing: drawing paper, drawing board, a needle for marking points, pencil, etc. Then, practical hints for drawing are given, it is shown how to check a ruler on its straightness, a right angle to be rectangular. In this manner, always preceded by actual drawing and through lively intuition, the doctrine of simple planar and three-dimensional configurations is developed in a quasi purely empirical manner. Going a little further than this very basic book is Harrison and Baxandall's, "Practical plane and solid geometry for advanced students including graphic statics"[111] that leads, in the same empirical manner, to descriptive geometry and methods of graphical calculation. More references can be found in the very interesting report "*Über Reorganisationsbestrebungen des mathematischen Elementarunterrichts in England*" by Robert Fricke,[112] in which the Perry movement is discussed in detail. Also quite stimulating are the reports of the discussions which Perry organised at the Glasgow and Johannesburg meetings (1901 and
[235] 1905) of the British Association[113] – the English analogue of our German Meeting of Natural Scientists, and as a result of which he achieved a considerable impact upon school teaching in England.

I consider these teaching proposals by Perry certainly as very suitable *for in-service schools*, *and lower and middle vocational schools* which have to train practically competent craftsmen and low-level technicians. But for secondary schools, the exclusive emphasis on the practical characteristic of Perry's direction, is to my mind *not sufficient* although they certainly provide praiseworthy suggestions. One will not want to omit so completely the formation of logical thinking obtained by teaching mathematics. What is desirable will be some *middle course* between the two possible extremes: where along with the intuitive development of geometry, starting from practical experiences, the logical demonstrations will not be neglected.

Due to the pressure from the Perry-movement, the examination authorities in Oxford and Cambridge appear in fact currently to accept such a compromise, as recent exam regulations show.[114] The new textbook by Charles Godfrey and Arthur W. Siddons follows these new tendencies: "Elementary geometry practical and theoretical",[115] which reveals considerable progress compared to the textbook of the

[110] London 1903.
[111] London 1903.
[112] About reorganisation efforts of mathematical elementary education in England. In: *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 13, pp. 283 sqq., 1904.
[113] *Perry:* Discussion on the teaching of mathematics. London 1902. – Discussion at Johannesburg on the teaching of elementary mechanics. London 1906.
[114] Regulations of the Oxford and Cambridge Schools Examination Board for the year 1904; on p. 37, there is a proper section on "Practical Geometry".
[115] Cambridge 1904.

Association. It starts with an *Introduction appropriate for intuition* ("experimental geometry") for the first stage, a geometric propaedeutics which in Germany has already for a long time been common practice – but in England hitherto barely known. Then follows *the logical development of geometry*, which reveals again close relations with Euclid in substance and in form, but is occasionally interspersed with new ideas. For instance, the *area* of a figure is first introduced almost so that one has to draw the figure on graph paper and counts the enclosed squares. This book, which one can probably regard as evidence for the eventual onset of a slow modernisation of teaching in England, has immediately proved enormously popular. Given the tremendous demand throughout the British colonial empire, one has clearly to reckon with completely different numbers when comparing the British book market with the German one.

Some Schoolbooks Considering the Requirements of Reform

It does not contradict the general conservative nature of the English school system that individual authors develop extremely free and interesting ideas about teaching, without intending directly to initiate an organisational change or to be able to **[236]** do so. As an example, I mention the new book by Benchara Branford, "A study of mathematical education, including the teaching of Arithmetic".[116] It contains very stimulating studies about the psychological conditions of teaching and it takes into consideration the parallelism that exists between the history development of the child and the history of mankind; the mathematical understanding of the child, which is addressed by the first teaching, comes thereby in parallel with the mathematics of indigenous peoples.

Moreover, I would like to mention "The first book of geometry" by Grace Chisholm and William Henry Young,[117] which was translated as "Der kleine Geometer" 3) by Sergej and Felix Bernstein.[118] Here, a new, original way is presented to guide the child into the understanding of geometry, and namely directly into three-dimensional spatial intuition. The guiding idea is that natural space intuition must slacken of necessity when one accustoms the child to draw exclusively from the beginning on two-dimensional paper and thus limits the child's intuition artificially to the plane. In this book, one operates from the beginning with the interesting tool of paper-folding, where alone, with the help of pins, all possible spatial and plane figures are formed. Here arise extremely vivid and yet at the same time logically satisfactory proofs, e.g., for the Pythagorean theorem; and it creates a new, more interesting teaching structure of geometry, relevant even for the higher levels of teaching.

We leave the English situation and turn to France.

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[116] Oxford 1908 [German translation by *Rudolf Schimmack and Heinrich Weinreich.* Leipzig 1913].
[117] London 1905.
[118] Leipzig & Berlin 1908.

II. The Teaching in France

Petrus Ramus and Clairaut

The conditions here are the more interesting for us since they have influenced in various ways the developments in Germany. Here, a situation fundamentally different from England is revealed. While the strictly conservative Englishman adheres to the old institutions, the Frenchman loves the new and achieves it even if often – rather than by continuous transformation of the Old – by *sudden* reformation, which somewhat constitutes a revolution. The organisation of teaching is entirely different: in France, there is not only centralisation of the exam – due to entrance examinations to higher education institutions, especially those in Paris – but also generally a strictly centralised organisation of teaching. The supreme authority, the *Conseil d'Instruction Supérieure* (amongst its members being always mathematicians of the first rank) is the absolute ruler and is entitled to decree, at its discretion, far-going reforms and changes as often as it wishes. Such reforms have to be re- **[237]** alised throughout the country immediately, and the teachers must see how to cope. The individual freedom to a high degree of each teacher, to which we in Germany are accustomed is less in practice here. One might even speak of a "system of revolution from above".

Now with regard specifically to the *teaching of geometry*, its *modernisation*, i.e., its liberation from the strict adherence to Euclid, began in France very early, at *about 1550* to give a round number. It is only one of the symptoms in the great struggle of the new humanism against the old scholasticism, which took place at that time. It was exactly then that *Petrus Ramus*, who occupied a prominent place among the representatives of the new ideas, not only for mathematics but also in other areas, wrote a textbook on mathematics ("Arithmeticae libri 2, geometricae libri 27"[119]). Ramus already completely abandoned Euclid's form and substance; rather, Ramus, as he characteristically says at the head of the first book, conceives of geometry as the art of measuring well ("ars bene metiendi"). Accordingly, the *practical interests* constitute the focus; he explained mainly how to perform simple geodetic measurements, describes the instruments and illustrates it all by numerous interesting figures. Logical deductions are exposed only as a secondary focus, but

[119] Basel 1580.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_15

by no means as an end in itself, but only to derive new propositions which cannot be obtained immediately by observation and are yet useful for applications; to be sure, deduction is not marginalised as much as with Perry.

This manner of teaching geometry was maintained in France for long time. About 200 years after Ramus, Clairaut published his "Eléments de géométrie".[120] This is the same Clairaut who is well-known as an outstanding researcher; in fact, we can remark that in France, unlike in Germany and England, there were always important university mathematicians who devoted themselves to issues of mathematics teaching. Clairaut's work is characterised by its excellent style. Actually, the French are masters of highly readable expositions, even of more difficult and abstract issues, which constitutes the sharpest contrast to the uniform, stereotyped "Euclidean" style. Such books read "like a novel" and therefore refute most strikingly the old view, good books on science have to be written in a boring style. Now as regards the *content*, Clairaut starts from *practical problems of field surveying*
[238] and leads gradually to general ideas, where the strict logical moment stands somewhat back. He argues in his very interesting preface, why he chose this approach: the practical problems of surveying incited mankind to develop a geometrical science; therefore, if one begins the book with them, one will succeed in interesting anybody much more in geometry – than by an abstract building of axioms and theorems, whose internal meaning nobody can easily understand. Clairaut's tendency was obviously to make his work accessible also for a wider, not properly professional public; actually, mathematics then served to a much higher degree in the general education of the ruling classes than is the case today.[121]

Legendre's *Éléments* and Their Importance

A new era in the organisation of teaching occurred at the end of the century in the wake of the great upheavals following the French Revolution of 1789. While hitherto it had essentially concerned only the education of the upper classes, and

[120] Paris 1741. [Translator's note: This book was often re-edited and also translated in various languages.]

[121] [Translator's note: Many mathematicians interested in school teaching have expressed favourable opinions of Clairaut's textbooks on geometry and algebra. Klein might have been particularly attracted, since Clairaut seemed to refer in his preface to a use of history in teaching. Actually, he did not analyse history but imagined how first mathematicians might have invented the first notions, which would have been supposedly simple – and thus apt to be used in teaching for beginners. Klein's presentation shows him aware that it was directed to the nobility. Glaeser showed that this textbook for leisure time at best constitutes a problem-oriented approach (Georges Glaeser, À propos de la pédagogie de Clairaut, *Recherches en didactique des mathématiques*, 1983, 4: 332–344). D'Alembert had already criticised Clairaut for not developing geometry systematically. The book did not present proofs and proving. Lacroix, who wanted to use Clairaut's algebra for the new public school system from 1795 on, had to abandon it, learning that this style was not adapted for a system of public education (see Gert Schubring: *Análise Histórica de Livros de Matemática. Notas de Aula.* Campinas: Editora Autores Associados, 2003).

specifically the training for an officer's career, now it became essential for the new social layers of the middle classes. Teaching was organised according to new goals and methods. I have to emphasise two directions of development, which start from the two then recently founded Parisian higher education institutions: the *École polytechnique* and the *École normale supérieure*. The former was created - due to the new growth in technology – for the training of engineers and military officers, the other was intended for the training of teachers at secondary schools.[122] At the École polytechnique the most influential man was the famous *Gaspard Monge*; he created there the facilities for teaching geometry, such as we still have them today at the technical colleges and similar institutions, especially the major lecture course on descriptive geometry and analytic geometry. The substantial change from the previous organisation was the fact that not only were a few particularly interested students especially stimulated, but that, at the same time, a large number of students were fruitfully guided in their own work by means of appropriate organisation. It made an enormous impression on Monge's contemporaries when he guided classes for the first time, in which about 70 students were busy at their drawing boards.

On the other hand, Adrien-Marie Legendre taught at the *École normale*;[123] for a long time, he exercised a dominant influence on geometry teaching by means of his famous textbook *Éléments de géométrie*, published originally in 1794. I can show you here the 4th edition of this book.[124] This work was, next to Euclid's Elements, the most widely disseminated of all textbooks on elementary geometry, and remarkably – as I indicated already – not only in France, where it was repeatedly **[239]** reprinted throughout the 19th century, but also in other countries. In America and Italy especially it held a dominant position for a long time.

In comparison with Clairaut, or even Petrus Ramus, Legendre's book represented a *major step towards Euclid*; his main goal was once again a self-contained abstract system of elementary geometry. On the other hand, there are *substantial differences compared to Euclid*, which I shall now explain in detail in view of its great historical significance:

1. First, with regard to the style of presentation, Legendre provides a coherent, easily readable text; in its outer form, it approaches much more Clairaut's style, which I have just praised, rather than Euclid's style characterised, as is known, by its – let me say – chopped manner, fatiguing in its monotony.

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122 [Note of the translator: The latter is a misinterpretation, which one frequently finds in the literature even today: the *École normale* of the year III was created in 1795 to train teachers for the primary schools in only four months, according to the *méthode révolutionnaire*, in order to inaugurate the first system of public education. The *École normale supérieure* was created in 1808/1810, as a part of the *Universite Impériale*, in fact, for the formation of secondary school teachers.]

123 [Note of the translator: This is information, which one still encounters today in many publications. Yet, Legendre did not teach at the *École normale*. In the 1770s, Legendre taught only for a short time at a military school, the *École Militaire*, but never afterwards.]

124 Paris 1802.

2. With regard to the *content,* probably the most essential point is that Legendre makes *conscious use* in geometry, contrary to Euclid, *of the elementary arithmetic of his time*; he is therefore – to use these terms – an adherent of the *fusion of arithmetic and geometry* and even brings trigonometry into this fusion, for he also treats this in his textbook.

3. The *principal position* of Legendre, as compared to Euclid's, is *shifted somewhat from the logical side to that of intuition*. Euclid puts all his emphasis – I have stressed this often enough – on the structure of logical deductions; he keeps it free – at least in principle – from the interference of elements of intuition. All that he thinks necessary as facts of intuition, he arranged beforehand in his axioms, etc. In contrast, Legendre does not avoid sometimes using intuitive considerations within his deductions of geometrical theorems.

4. To go into more detail, it is particularly interesting to compare the *treatment of irrational numbers* by both authors. In Euclid's Book 5, the concept of irrational number is defined – as we know – extensively in the form of logos or ratios of two incommensurable quantities and investigated in complete analogy with the modern theory of irrational numbers. Later on, Euclid provides the proofs of those theorems which concern – according to the essence of the matter – irrational numbers, particularly carefully and with a rigour almost sufficient for modern requirements (proof by the exhaustion principle!). Legendre, however, glides quickly over all these issues. He assumes the numbers, rational and irrational, as known from arithmetic;[125] then, of course, one did not bother much at that time about its rigorous foundation
[240] And he does not know proofs by exhaustion and the like; it is to him entirely evident, without any further explanation, that a theorem valid for all rational numbers holds also for all irrational ones. Here, too, he is in agreement with all the other great mathematicians of his time. I presented only last semester an example of this viewpoint from Lagrange's "théorie des fonctions analytiques".[126]

5. Despite these less rigid positions of Legendre regarding the logical rigour of individual expositions, he is not at all indifferent concerning the principal issues of the foundations of geometry; unlike his predecessors in France, he not only builds on the Euclidean tradition with full interest, but he even promotes it by essentially new ideas.

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[125] [Translator's note: As the following paragraphs show, Klein was aware that later editions had changed Legendre's text. In fact, after Legendre's death, his twelth edition was re-edited until 1840. But in 1845, the publisher asked an otherwise unknown person, Alphonse Blanchet, to edit a new version; these editions decidedly changed Legendre's conceptions. And it was, in particular, Legendre's careful distinction between proofs for commensurable and for non-commensurable quantities, which were abolished by Blanchet. See for instance the two separate proofs XVI and XVII in Legendre's original book II, and the proof XVIII in Blanchet's book II. See: Gert Schubring, "La diffusion internationale de la géométrie de Legendre: différentes visions des mathématiques", *Raisons – Comparaisons – Éducations. La Revue française d'éducation comparée*, 2007, *2*: 31–54. One wonders why Klein had not compared this particular issue, while he compared another issue in the two so strongly different textbooks (see p. [242]).]

[126] See Vol. I, p. [165] (proof of the binomial theorem).

Excursus on Legendre's Theory of Parallels

It is to the *theory of parallels*, which he especially directs his attention, and I should like to comment on this in more detail. Incidentally, one must study this in the early editions, since the later editors have changed much, especially on this issue.

I start from the following observation: we had earlier characterised the Euclidean and both the non-Euclidean geometries by the fact that the number of straight lines through a point parallel to a given straight line is either 1 or zero or [at least] 2; Instead, one can, however, refer to the *sum of the angles of any rectilinear triangle* and obtains the following distinction, which turns out to be exactly equivalent to the previous one, as one can show: In Euclidean geometry the sum of angles is π, in the non-Euclidean of first type (hyperbolic) it is always less than π and in that of the second kind (the elliptical) it is always greater than π. Now Legendre wants to prove that the last two possibilities are excluded. Since that is nothing else than to prove the Euclidean parallel axiom, it can only be achieved by borrowing from intuition certain simple basic propositions which imply the parallel axiom; his mastery was now to select these in such a plausible manner that the reader and certainly also the author himself did not realise that he had indeed imposed new restrictive conditions.

With respect to, at first, the impossibility of *elliptical geometry*, that is the sum of the angles $> \pi$, Legendre's proof - a highly remarkable one – depends upon the tacit *assumption of the infinite length of a straight line*. This is certainly a very plausible assumption – and neither Legendre nor any of his readers will have doubted its correctness. In fact, all geometers before Riemann have regarded it as evident. And yet, elliptical geometry shows that the assumption of straight line of finite length is compatible with the other axioms – if only one assumes it as *unlimited*, as returning back into itself. Thus, one has to be aware that one derives a new and decisive fact from intuition concerning the infinite length of the straight line.

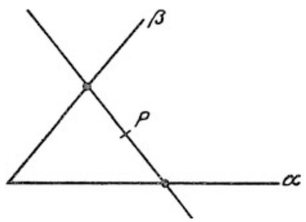

Figure 142

To also exclude the hyperbolic geometry, Legendre used likewise – without special mention – a simple fact of intuition which no naive mind, so to speak, nor one miseducated by geometric studies will ever doubt: If P is any point inside the angle of two half-rays α, β, then one should always be able to draw a straight line through P that meets both α and β (see Fig. 142). By using this condition, he unobjectionably succeeds in proving that the angle sum in a triangle can never be less than π; thus, finally, the Euclidean geometry alone remains as valid. **[241]**

I must now explain *in how far this so plausible fact does not apply in the case of non-Euclidean geometry of the first kind*; only then can we understand completely that Legendre succeeded by its use in excluding this geometry. We start from our earlier observations (p. [198]). Let α, β be two rays of hyperbolic geometry through the point *0*, which has of course to lie inside the fundamental conic section $\Phi = 0$ (see Fig. 143). All the parallels to α are then the rays through the intersection of α with the conic section (i.e., the point at infinity of α), as far as they lie in the interior, and the analogous applies to β. Thus, there is a straight line γ, which is parallel both to α and to β, namely the connection line of their intersections with the conic sections $\Phi = 0$. Of course, this cannot happen in Euclidean geometry. If we choose now the point P between α and β outside of the triangle delimited by γ with α and β (but within the conic section), then Legendre's assumption is no longer valid, because each straight line through P will meet only one of the rays α, β within the conic section, but the other outside, that is in the sense of our geometry: it will not meet the straight line at all. And that I wanted to show here.

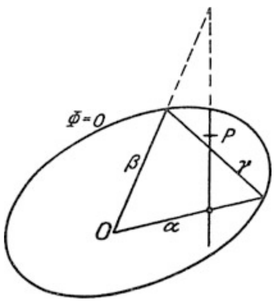

Figure 143

Legendre's Successors

After this excursus, let us now leave Legendre and observe how *geometry teaching developed in France after him.* Strangely enough, the organisation of the school system in France changed very little in the course of the 19th century; actually, in general, in all cultural areas and for a long time the institutions created under Napoleon I outlasted all changes of the political regime. Thus, in the teaching of geometry, Legendre continued to dominate almost unlimitedly; only that in the
[242] many new editions[127] there occurs a certain filtering of the content in the sense of limiting the relations with the applications that are still present in Legendre. While Legendre himself no longer gives to the art of geometric measuring the same

[127] I have here at hand the 33rd edition, edited by Alphonse Blanchet, Paris 1893. [Note of the translator: See my earlier note: Blanchet's altered editions of Legendre, from 1845 on, have a separate numeration; at least, Blanchet was so honest not to count them as direct continuations of Legendre's original.]

excellent role as Clairaut or even Petrus Ramus, he does not reveal the disdain for it which arose later on: moreover, the esteem for mathematical *execution*, and for numerical computation is still quite characteristic of his editions. But everything related to this is omitted more and more in the later editions: in particular, the chapter on trigonometry in which Legendre had revealed a close relationship with those applications is omitted. As a characteristic example I will mention the so-called Legendre theorem of spherical trigonometry. If one has a spherical triangle with sides *a, b, c*, and the angles α, β, γ on the spherical surface (see Fig. 144), the so-called spherical excess $\alpha + \beta + \gamma - \pi = E$ is always positive, as is well-known. Now, if the sides are not too great in relation to the radius of the sphere, for example, on the earth's surface not greater than 100 km, one can replace the spherical triangle, with an accuracy sufficient for all practical purposes, by a flat triangle with the angles

$$\alpha - \frac{E}{3}, \beta - \frac{E}{3}, \gamma - \frac{E}{3}.$$

Legendre proves this nice theorem, which is actually used a lot in geodesic practice, very easily, by using, in the formulas of spherical trigonometry, only the first terms in the series for the trigonometric formulas. In the later editions of Legendre's book you will search in vain for this theorem.

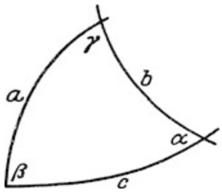

Figure 144

In addition, the later editions of Legendre demonstrated another tendency, which is characterised by the extensive "Traité de géométrie" by *Eugène Rouché* and *Charles de Comberousse*.[128] In France, the mathematics teaching preceding the studies in higher education is much more demanding than in Germany. The transition to higher education is prepared for by a two-year course called *Classes de Mathématiques spéciales* in which there are given no less than 16 weekly hours to mathematics; all who will later need to use mathematics obtain in these an extensive grounding. Due to this structure, there arose the need, to add *a lot of new material* to the textbooks of elementary geometry, and this is what is done typically in the *Traité* by Rouché-Comberousse, which proved a very popular textbook. It contains in numerous notes expositions on non-Euclidean geometry, triangle geometry, tetrahedral geometry, the study of the more important curves and surfaces, and much more.

[128] Parts 1 and 2, sixth edition, Paris 1891.

[243] The Reform of 1902

I am now turning to the new reform movement in mathematics education, initiated in France from 1900 and quite analogous to our German reform efforts. Again we can bring this movement in relation to shifts in the entire cultural image of the era. Because of the tremendous upsurge of trade and transport, technology and industry, an urgent need for participation in all cultural achievements, and for education in all fields, not least in mathematics, arose in ever wider strata of the population; admittedly, the emphasis is not on theoretical interests, but the aspiration for useful, immediately applicable practical skills. One should by no means accuse the leaders of this movement of being devoted to base utilitarian aims; since it is a high and ambitious goal – raising the general professional competence – that they are envisaging.

It is characteristic of the French situation, that this reform began with deliberations in the Paris Chamber of Deputies; a Commission created there, after consulting a large number of public bodies, elaborated a detailed report about a comprehensive reform of the secondary school curriculum; in which, the reform of mathematics teaching is one important element in a long chain of issues. The main aspects of its reform are simplification and greater intuitiveness of teaching, on the one hand, and, on the other hand, the transfer of certain issues into the school curriculum, which used to be attributed from immemorial time to higher mathematics and which are not only easily accessible, but also of the utmost importance for the modern cultural life, especially for science and technology. I mean the concept of function, graphical representation, the elements of the infinitesimal calculus. One aspires, in particular, to a much closer connection between arithmetic and geometry than one ever sought before – it means the apogee of fusion in the widest sense. This reform has been formulated in the *Plan d'études* of 1902[129] and was immediately generally introduced. This uniform procedure evidences the effect of the previously mentioned extensive centralisation of the school administration in France: such a far-reaching reform necessitates only a decree by the highest authority. This entire development is dealt with in detail in the volume of my "*Vorträge über den mathematischen Unterricht an den höheren Schulen*"[130] edited by Mr. Schimmack, which I am recommending to you. You will find there much information on the
[244] organisation and development of mathematics teaching in general that complement and completes what was written here specifically concerning geometry. As for the new French curricula, I want to emphasise at this point only that the old elementary geometry in the Euclidean sense is now quite strongly reduced in favour of the modern new ideas. You will find this confirmed if you look into one of the most important textbooks based on the new curricula: the *Géométrie* by Émile Borel;[131] it is

[129] Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons. Paris 1903.

[130] Lectures on mathematics teaching in secondary schools. Part I. Leipzig 1907.

[131] Paris 1905. German translation, as Elemente der Mathematik, in two volumes, by Paul Stäckel. Leipzig 1909. [Second edition: Volume I 1919, Volume II 1920.]

a very interesting book in which the subject matter is arranged in an easy and manifest manner; by the way, the practical interests abound in it immensely strongly.

In contrast, it is remarkable that the French curriculum also reveals now renewed interest in a completely logically elaborated teaching structure of elementary geometry. I want to draw your attention especially to a very significant book, the "Nouveaux éléments de géométrie" by Charles Méray in Dijon, which, although first published in 1874, attracted the attention of a wider circle only in recent years.[132] Méray used in his proofs not a single fact of intuition, which he had not formulated previously as an axiom, and develops in this manner a *complete system of axioms for geometry*. He succeeds in satisfying the requirements of the actual teaching far more than the strict followers of Euclid, since he does not strive for reducing the number of axioms strictly to a minimum of independent propositions; moreover, he formulates them in general only when he really needs them. Especially characteristic of Méray is, first, that he realises the fusion of plane geometry and solid geometry as completely as possible, and secondly, that he, unlike Euclid puts the concept of motion group at the head and consistently bases his entire structure of geometry on it. He realises thus a foundation of geometry, which is very similar to the one as we outlined recently: translations and rotations are from the very beginning complementary elements; the former provides the concept of parallelism, the other – since they concern space from the outset – the concept of rotation about axes perpendicular to the plane, in which the path curves (circles) of each point lie. You may read yourself the very interesting exact implementation of this structure in Méray's book. I mention here only that he always paid special attention to the exact realisation of all the necessary limit processes. For this aim, he sometimes uses the modern number concept in its rigorous formulation – although he does not go as far in the fusion with arithmetic and with analytic geometry, as **[245]**
we did it here.

By the way, you can see clearly in the modern French schoolbooks the influence of Méray's approaches. For example, in Borel's book already mentioned, the concept of motion is very important, and even more so in the new "Eléments de géométrie" by Carlo Bourlet,[133] who is the author of many very common textbooks; there the group of motions and the geometrical quantities of their invariants are everywhere explicitly used. We are thus leaving France and now going to Italy.

[132] Nouvelle édition Dijon 1903; 3ème édition 1906.
[133] Paris 1908.

III. The Teaching in Italy

The Influence of Cremona

In Italy, we note another highly characteristic development that reveals quite different patterns than those in England and France; in their typical forms it can at the extreme be placed in parallel with Méray. I want to concern myself only with *modern Italy from about 1860* onwards. The greatest influence on the uniform restructuring of mathematics teaching in the then newly unified state was Luigi Cremona, the same person whom you all know for his scientific importance in the development of modern geometry; actually, he is the *founder of the independent algebraic-geometrical research* in Italy, which has provided such excellent results. Due to his scientific activity, Cremona has exerted a lasting impact on *higher education*, by connecting *projective geometry with descriptive geometry and graphical statics*. Engineers everywhere in the world speak today of *Cremona's force diagram*, and if this name may be historically unjustified, it shows clearly Cremona's great influence.

Strangely enough, Cremona had an effect on the teaching at secondary schools in a very different sense. In a famous expert opinion of 1867 he recommended *Euclid*, if not obligatory, but mainly to introduce this as an exemplary textbook for all geometry teaching at schools, because of its arrangement and limitation of the subject matter and, in particular, in its ideal of a strictly logical closed structure of geometry. Thus, Cremona emphasised especially the logical side, while – in his own teaching activity and in his scientific work – it was mainly the intuitive moments which prevail.[134] It is difficult to understand what actually constituted the link between Cremona's two apparently so strongly conflicting objectives.

Older Geometry Textbooks [246]

In any case, Cremona's suggestion of 1867 fell on extremely *fertile ground*: and the Italian mathematicians developed a true rivalry to replace Euclid by textbooks

[134] See Cremona's Elements of projective geometry, 1872. In German by Trautvetter, Stuttgart 1882.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_16

which retained its subject matter and its entire tendency but which realised these in a manner corresponding more to today's tightened requirements. It is characteristic that a number of scientifically eminent mathematicians participated in this work, just as we saw was the case of France, and that, accordingly, a number of scientifically very important works were published – whose educational value one, however, does not necessarily rate as highly. A very interesting report about the most important manifestations of this movement was published by Walther Lietzmann;[135] I want to emphasise in the following, partly in connection with this report, a few particularly characteristic moments.

I begin by commenting on the *Euclid translation*[136] organised by Enrico Betti and Francesco Brioschi in 1867 and which initiated the dissemination of knowledge on Euclid in Italy; it contains, like the English school editions of Euclid, only the books 1 to 6, and 11 and 12. However, unlike the English tradition, their tendency is by no means to present the subject matter in this ancient form: they just aim to provide the base for working in an independent scientific and pedagogical manner. Among the textbooks written in this way, a greater number of the older ones still remain as close as possible to the Euclidean schema of definitions, etc.; however, all the numerous facts drawn from intuition, which Euclid tacitly uses, are formulated explicitly and exactly. To fill the gaps in the first book, one counts, according to general opinion, also the concept of rigid motion among the concepts tacitly applied by Euclid; therefore, this concept is established as the basis of the system, by formulating a number of "axioms of motion". Much like Méray, no emphasis is placed on a mutual independence of the individually established axioms, for pedagogical reasons. A typical book of this tendency is the very popular "Elementi di geometria" by Achille Sannia and Enrico d'Ovidio,[137] published first in 1869 in which you will find all the above comments confirmed. The subject matter is just as limited as in Euclid, only it is presented in a considerably smoother form. For example, the number concept of pure arithmetic is avoided by all means, but the
[247] idea of the limit concept, which is underlying the Euclidean proofs by the exhaustion method, is more clearly elaborated than in Euclid. Furthermore, in particular, planimetry and stereometry are separated externally, but apparently it is expected that the book could be used in schools with a "fusionist" curriculum, since the efforts for fusion between planimetry and stereometry are especially widespread in Italy. As a textbook, which promoted the fusion movement the most, I want to mention the "Elementi di geometria" by Riccardo de Paolis.[138]

[135] Walther Lietzmann, Die Grundlagen der Geometrie im Unterricht (mit besonerer Berücksichtigung Italiens). *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, vol. 39, pp. 177 sqq.
[136] *Gli elementi di Euclide*, 36a Ristampa. Firenze 1901.
[137] Vol. I, Vol. II, 11th edition, Napoli 1904.
[138] Torino 1887.

New Demands for Increased Rigour; Veronese

Much more than this and similarly related textbooks *another group of textbooks* moves further from the Euclidean representation by striving to achieve a much *higher degree of rigour* in their elaboration of the fundamentals. They judge the numerous geometric basic concepts of Euclid and of those textbooks just described as not sufficiently well defined and therefore aim to proceed with *one single basic concept*, namely, of the *point,* from which all the other configurations needed in geometry should be constructed purely logically. In particular, any use of the concept of rigid *motion* should be *entirely avoided* in the foundations of geometry.

The culmination of this development is probably represented by the various textbooks of Giuseppe Veronese, which cover the entire field of geometry. Relevant for us here are not his "*Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten geradliniger Einheiten, in elementarer Form entwickelt*",[139] which is by no means a textbook, but a treatment in the most abstract form of the purely scientific problem of a general multi-dimensional and "non-Archimedean" geometry; rather we are interested in his textbooks "Nozioni elementari di geometria intuitiva"[140] and "Elementi di geometria".[141] The first is an inductive introduction, which makes the student of the lower level of secondary schools familiar with the various geometric forms – it corresponds somewhat to our propaedeutic *Vorkursus.* The actual systematic geometry teaching begins namely, according to all Italian curricula, only *very late*, in grades that correspond to our two *Sekunda*. One should therefore not believe that all these exact textbooks address themselves to boys of the age of students in our *Quarta*!

The "Elementi" of Veronese provides the *theoretical developments* in an extremely complete manner; all postulates of geometry are enunciated – even if they appear as ever so evident; for example, it is explicitly stated as a first postulate: "There are *several* points" – this means that we do not consider a geometry in **[248]** which there exists only 1 point! Here, incidentally, one still refers, at least briefly, to the empirical observation that is conductive as a heuristic principle for the establishment of axioms. Specifically, Veronese used the straight line as the fundamental geometric configuration, which he defined as a system of points satisfying certain requirements. The congruence of such segments then serves as a foundational concept to which everything else is reduced – in a very original way, therefore, two *triangles* are called congruent if all sides are congruent, and this implies the definition of the congruence of angles (i.e., the third theorem of congruence precedes!). Even the *theory of parallels* is presented in this way: two straight line are called parallel, when they lie centrosymmetrically with respect to a point; i.e., all straight lines through this point cut segments of pairwise equal length. By the way, Veronese also maintains his textbook *within the limits of Euclidean subject matter;* in partic-

[139] In German translation by A. Schepp, Leipzig 1894.
[140] second edition Verona 1902.
[141] There are various editions: for example, Ad uso dei ginnasi e licei. Con collaborazione di *P. Gazzaniga.* P. 1., II. 3. Auflage. Verona 1904.

ular, he evidently avoids any reference to arithmetic. Allied to the Veronese book with regard to the contents are the "*Elementi di geometria*" by *Federigo Enriques and Ugo Amaldi*,[142] who, however, stress, the pedagogical considerations to a much higher degree, in addition to the rigorous systematics.

The Peano-School

Veronese's abstract direction has now even experienced an increase through the work of the so-called *Peano-school*. Giuseppe Peano in Turin represents the tendency to realise the purely logical presentation of mathematics, free of any element of intuition in a much more radical form than was aimed at in the axiomatic investigations on which I have previously commented. To this end; Peano has invented a special *formula language*,[143] which should replace ordinary language. He thinks, namely, that otherwise one could not at all fail to be affected by non-logical factors, due to the innumerable associations that spontaneously attach themselves to words familiar to us. Hence, the ideal becomes finally to *operate with meaningless symbols according to "arbitrary" rules* that themselves per se also mean nothing. Peano created a large school in Italy, which is widespread now and has a lot of influence. Together with his disciples, he published a "Formulaire", in which all of mathematics should be presented in its purely logical content according to his formula language.

If we ask whether such an extreme emphasis on purely logical considerations can be beneficial for science, I like to apply a parable: many people appreciate, when they climb from a valley up to a mountain, into the purer and thinner air, and yet it is not at all the case that an ever increasing thinning of air always increases the well-
[249] being; there exists a limit beyond which even any possibility of life ends. Thus, I believe that the enthusiasm of the logician for the elimination of any intuition (if that should ever be possible, since the Peano symbols as such entail a remainder of intuitive elements in his system!) is somewhat premature: while some might at first appreciate this purer logic, there will exist also here an optimum in the distribution between logic and intuition, when grounds in favour of the former cannot be trespassed without damage!

Admittedly, in terms of *pure research*, one will of course approve any new approach and await the progress and suggestions that it will yield. But it is necessary to assess it also from a *pedagogical* viewpoint because such abstract tendencies seem to have often achieved an influence on school teaching. There this judgment will have a basically negative result: one can rightly assume that teaching according to this tendency will mean that many students learn nothing and that the few who will be able to follow at all will surely not receive anything they can use later.

[142] Second edition, Bologna 1905.
[143] [See about it also: Vol. I, p. [13] & [286].]

In fact, a *reaction* against this too abstract manner of teaching seems to have emerged in Italy – also in higher education because, strangely enough, the pure logicians have often attained a dominance just at the *technical colleges*. One now complains there about the poor mathematical education of the average of students who cannot understand abstract arguments. Years ago, I was sometimes told as an interesting example of a lack of adaptation to the real needs, that in lecture courses for engineers one proved Taylor's theorem first for any number of variables, and only later specialised it for one variable.

Efforts for Reform

Even in *secondary school teaching* reform efforts have more recently tended to become active. As with our German and the French movements they seek to abandon the predominant consideration of abstract logic and the close adherence of the subject matter in Euclid's *Elements*, and instead enliven mathematics teaching by *intuitive moments*, by integrating the major general concepts of modern science (the *function concept*), and finally by *including applications*. The leader of this movement is Gino Loria, who reported at the 3rd International Congress of Mathematicians in Heidelberg, in 1904, about mathematics teaching in Italy[144] and thereafter argued in an interesting lecture, also translated into German,[145] at the meeting of "Mathesis", the Italian Association of Mathematics Teachers, about his **[250]**
reform proposals. This Association is a testimony that modern ideas have now become of key interest for teachers in Italy. Even though the new curricula of 1905[146] reveal only few traces of this, we may assume perhaps that gradually Italian schools will be freed from the chains of extreme logic and will introduce a newer form of teaching.

We now finally turn to our own country.

[144] Verhandlungen des 3. internationalen Mathematiker-Kongresses, p. 594. Leipzig 1905.
[145] "Vergangene und künftige Lehrpläne." German translation by *H. Wieleitner*. Leipzig 1906.
[146] Istruzioni e programmi vigenti nei ginnasi e licei. Torino 1905.

IV. The Teaching in Germany

The Influence of Primary School Teaching (Pestalozzi and Herbart)

In principle, I want to consider all the German-speaking countries, such as German-speaking Switzerland and Austria. In Germany, the ways in which the teaching of geometry have evolved show completely different patterns to those in the other countries; especially, due to the lack of *uniformity*, as it was achieved in other countries – be it by strict governmental organisation or by the intervention of a strong personality. Here in Germany public education became established in each individual state according to proper conceptions; moreover, also at the individual institution, for individual teachers always retained a relatively large degree of freedom for independent practice. Thus, a great number of various suggestions from different sources achieved realisation concurrently; usually, their efficiency could be established, even before they had been sanctioned in official curricula. I shall be able, of course, to select out just a few aspects which became particularly important for the development in the last decades – say from about 1870 on. For additional information, I recommend to you the extensive presentation of the general lines of development in the book Klein-Schimmack.[147]

A particularly important tendency that has become influential since the seventies was occasioned by movements in primary education – in connection with the increased need for education of broad strata of the population and in connection with the national upswing at that time. It is the conception that elementary teaching must necessarily be based on immediate intuition, that teaching there always has to relate to visible things, familiar to the pupils. These conceptions stem, as is well-known, from the famous Swiss Heinrich Pestalozzi, whom one can see as the founder of elementary education in the modern sense. His period of activity was – in round numbers – around the year 1800. Certainly, it is of interest for any mathematician to know Pestalozzi's original papers, which are pertinent for mathematics. These are "Das *A B C der Anschauung oder die Anschauungslehre der Maßverhältnisse*"[148] and "*Die Anschauungslehre der Zahlenverhältnisse*".[149] These books **[251]**

[147] Quoted on p. [243].

[148] The ABC of intuition or doctrine of intuition of measuring relations. In two booklets, Zürich and Tübingen, 1803.

[149] The doctrine of intuition for numer relation. In three booklets, Zürich and Tübingen, 1803/04.

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5_17

are intended to show how a pupil with no previous education at all can succeed in obtaining access to the simplest facts of space and number intuition. Of course, one would be quite mistaken if one would expect something particularly thrilling in them; actually, they present almost the most boring thing I've ever had in hand: these books try only to present all possible trivial relationships with an extensively frightening consistency throughout. To give just one example: the child should learn that a square can be divided by horizontal and vertical lines into equal parts (see Fig. 145). To achieve this, Pestalozzi not only provides a table with all 100 combinations of divisions by 0, 1 ..., 9 in vertical and horizontal lines, but he also presents in the text also the number, position, etc. of the partial squares and rectangles in each individual case, always by the same scheme and in the most detailed manner possible.

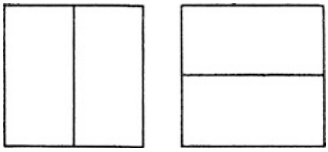

Figure 145

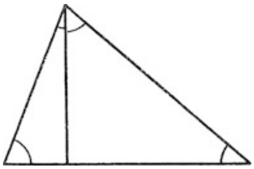

Figure 146

One has probably to understand that Pestalozzi thus aimed to provide even the clumsiest elementary school teacher – he then had still to count on quite inadequately prepared teaching personnel – with a rich collection of examples, from which they select any part and base their classes literally on that.

As a supplement, I am showing you here a booklet by the Göttingen philosopher *Johann Friedrich Herbart*, who was especially active in disseminating Pestalozzi's ideas: "*Pestalozzis Idee eines A B C der Anschauung*".[150] Here, Pestalozzi's approaches are continued in a less schematic and therefore more interesting presentation. Notably there are all kinds of triangular forms, which Herbart wants to make clear to the children. Thus, he presents in a table the triangle angles as well as the angles to the right and to the left of the altitude, from 5° to 5° (see Fig. 146) and in another table the associated lengths of triangle sides – with the intention that the pupils should verify this table by re-measuring the lengths. His proposal is similarly strange, that one should hold tables before the eyes of children already in the cradle:

[150] Pestalozzi's idea of an A B C of intuition. Göttingen 1802.

tables with the most diverse triangle forms, and ask them to engrave the different triangle forms in their in minds.

Pestalozzi and Herbart exerted a powerful influence on primary education, which even today continues. You will find in most textbooks on space doctrine [*Raumlehre*] significant impacts of Pestalozzi's ideas. In a very characteristic form you can find Pestalozzi's Anschauungslehre in our Kindergarten whose conception goes back either to him or to *Friedrich Fröbel*, where the children learn to know the simplest three-dimensional forms, by games with suitable objects. **[252]**

The Austrian Curriculum of Exner and Bonitz of 1849; Independent Emphasis on Space Intuition

Soon, these pedagogical ideas were received at secondary schools, too. Particularly characteristic in this respect is the curriculum, which *Franz Serafin Exner* and *Hermann Bonitz* elaborated around 1850 for Austria. That the movement began just there and at that time, can be understood again from the political situation; in Austria, the dogmatic method for mathematics teaching, dating from the Middle Ages, had been maintained, due to the many Catholic religious schools, especially those of the Jesuits; when the revolutionary movement of 1848 swept away the old, almost nothing of the existing could be used, so that the new became introduced in is purest form. Thus, the Exner-Bonitz curricula adapt, as far as possible, the new intuitive methods to the secondary schools. Space intuition becomes not only the method for the lower grades, as a preparation, but it turns out to become an *end it itself*. One should not just use intuitive subjects for exercising logical thinking, but the aim was to *exercise the intuition itself*. At the lower level (4 years), the recurrence of logic issues was entirely marginalised, and one practiced only the intuitive grasping of the figures by means of ever continued drawing. Also on the upper level, where the material thus obtained is subjected to logical consideration, drawing is maintained in a considerable extension. Many of you may have occasionally noticed how cleverly the Austrian mathematicians are able to draw – a consequence of that characteristic demand of the curriculum.

Transmission of These Tendencies to Northern Germany; the Textbooks by Holzmüller

These tendencies had begun, since the early seventies, to prevail in Prussia and in general in Northern Germany. Here we must be aware of the personal moment that Bonitz then entered the Prussian Ministry of Education, in an influential position. The guidelines for the reform were formulated for Prussia in the curricula of 1882. Its salient feature was the introduction of a *geometric Vorkursus*, the so-called *geometric propaedeutics* in the *Quinta*; here, the students should be made famil-

iar intuitively with subjects that should later constitute the content of the teaching structure of geometry. Compare, besides the book Klein-Schimmack, also my paper
[253] "*100 Jahre mathematischer Unterricht in den höheren preußischen Schulen*",[151] in which I aimed to describe the history of mathematics teaching in the last century.

The textbook, which probably realised the most marked formulation of the reform tendencies of 1882, is *Gustav Holzmüller's* "Methodisches Lehrbuch der Elementarmathematik".[152] Here the title is already characteristic: "methodical" is meant in contrast to "systematical"; it should not be established as a rigid teaching structure like Euclid had done, but a *natural* course of studies which is structured according to the experience how one will best foster the students. Moreover, it is not a textbook of geometry or of arithmetic, rather the entire *elementary* mathematics is presented: in alternating order of the individual subdisciplines, as they really can be taught in a consecutive manner in the classroom; even their mutual relations emerge clearly. Moreover, the geometric expositions are always based on real *drawing and constructing*. Particular emphasis is put on forming space intuition, on *stereometric drawing*. Attention is always given to understanding a construction, not only as a possibility, but that it is realised clearly and completely. The geometric theorems appear then often as side-results, one might say; for example, the theorems of congruence arise from the observation that the construction of a triangle, when three parts are given, is unique. I have to emphasise that the principles of projective geometry are partially integrated into the exposition, due to the tendency described. Of course, I cannot conceal that in Holzmüller's textbook the logical moments are at various times are treated too shortly; it is an old experience that one cannot satisfy all sides at the same time. When logic is stressed mainly, intuition will suffer, and vice versa.

The positive results of the efforts now described seem to have become meanwhile general teaching practice, but of course new ideas gradually entered. Primarily, as in all other countries, it is the strong movement that started in Germany around 1890 taken into consideration, which aims to strengthen the *applications of mathematics in all branches of the sciences*, particularly in technology, and its *importance for*
[254] *all aspects of human life*. It entails, in contrast to the tendency directed towards intuition something essentially new; while it is still possible to tie intuition with purely formal purposes, the issue is now actually to apply fruitfully mathematical thinking to the most different other areas. These efforts are closely related to the reform tendencies which he have discussed so often in the first volume of the present work and which I therefore only need to mention: the *introduction of the function concept, the graphical methods, the elements of infinitesimal calculus*, which all bring forward numerous new suggestions for teaching geometry.

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[151] "100 years of mathematics teaching in the Prussian secondary schools" In: *W. Lexis*, Die Reform des höheren Schulwesens in Preußen. Halle 1902. reprinted in: *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 13, pp. 347 sqq., 1904 and in *F. Klein* und *E. Riecke*, Neue Beiträge zur Frage des mathematischen und physikalischen Unterrichts an höheren Schulen, pp. 63 sqq. Leipzig 1904.

[152] In three parts. Leipzig (Teubner), 1894–95, and numerous re-editions.

Suggestions by Experimental Psychology

I will speak somewhat more extensively about some more recent further tendencies with which mathematicians have to grapple with more than has been done so far:

a) First, I mean since certain results of modern psychological research, especially of experimental psychology, and of modern hygiene. Herbart already tried to found pedagogy on psychology, but realising this objective has gained an entirely different basis, since psychology did establish exact experimental methods. Think, for example, how important is the study of *memory* for pedagogy, how important it is to know in which way facts are memorised and remain fixed in memory, and whether this depends on the environment or on the personal disposition of the individual. In fact, psychologists are busy with such research now, at many places, especially here in Göttingen. Similarly important for pedagogy is the study of *fatigue*, the question for example whether physical and mental fatigue are independent or not. Previously it was believed that one is particularly well fit for intellectual work after preceding physical effort, while now, based on observations, one generally accepts the opposite opinion.

A particularly important problem for psychology, particularly in view of mathematics, is the *difference of individual giftedness*. There was indeed a time when one was firmly convinced that only very few students are endowed with mathematical giftedness – this meant that only these were able to understand something of mathematics and that all the others could learn nothing, even upon applying the greatest efforts; the reason that such a view could find a so general acceptance can only be found in the defective method of mathematics teaching prevailing at that time. Later, when one began to value more the art of pedagogy, in the wake of the Exner-Bonitz curricula, one soon resolved to the opposite opinion that every student can learn, provided good will and some effort also from the side of the teacher, something considerable of mathematics. I expect from experimental- **[255]**
psychological research information on how this issue really has to be assessed. Certainly, there are also among otherwise gifted persons some who are quite "a-mathematical", who are absolutely not inclined for mathematical thinking. A recent very interesting conversation with the famous Berlin architect Messel showed me that such a-mathematicians occur also among artists of outstanding giftedness; Messel is known to all of you by the construction of the Wertheim department store, of likewise high convenient and artistic quality. When he heard that I am a mathematician, he spoke in the strongest terms against all that useless stuff, with which one is plagued at school – and which in any event had remained for him without any meaning. Perhaps it would be wiser if one would allow such characters through school without mathematics, as that all efforts are in vain to teach them at least some knowledge of mathematics. The only effect would just be to arouse in them a great aversion against these things that they do not succeed in understanding – and maybe with the effect to create influential enemies of mathematics. To be sure, that could only apply to the very few who are one-sidedly not mathematically gifted

while otherwise in excellent disposition; clearly, this is not a plea for smugness and laziness or for that old theory of the “general lack of mathematical giftedness”.

Other important tasks that await research by psychology on mathematics, refer to the undoubtedly existing finer differences between the types of mathematical giftedness, which prove their importance in the productive scientific workers, but which certainly are also relevant for questions of pedagogy. One encounters, on the one hand, mathematicians who are more predisposed for abstract arithmetic, while, on the other side, other mathematicians prefer to operate with geometrically intuitive forms. There are psychological investigations, in particular, on persons who have developed excellent skills in a narrowly restricted area, enormous calculators or chess players; also there, one found the greatest differences; one knows, for example, that some calculators see the large numbers, with which they operate, intuitively written in figures in their mind (*visual* giftedness), while others work in an auditory manner: by relating their associations with the tone of the number words. I refer in this regard to the interesting book by Alfred Binet, “Psychologie des grands calculateurs et joueurs d’echecs”.[153]

Relation to Modern Art Education

b) A second tendency, which manifested itself in modern times and that I should like to mention here, is related to what I just said about the mathematical giftedness of outstanding artists; I mean the modern so-called art education and innovations in modern drawing teaching. The goal is here, to achieve as soon as possible at a lively *intuitive conception of things as a whole*, and not to start with the study of their details, shows this effort, as it emerges in related manner with some excellent engineers, proves to be particularly interesting in the development of drawing teaching. Previously, the main stress was on the task that every student should learn to trace exactly specific contours according to templates - a method by which one too often generated little interest and little success. I remember that I always had to copy the same arabesque in my school days, since I did not all succeed in drawing this; thus certainly my ability to draw was not developed. Today, in contrast, one gives the child from the outset brush and paint in the hand and let it paint everyday objects according to its proper impression, as it has them immediately before their eyes or remembers them: accurate reproduction of details is not the task; the paintings can be entirely inexact provided the overall impression is successful. We see now everywhere in school exhibitions, which good results are obtained surprisingly with this method, even by children without any specific artistic talent.

Of course, this direction is in contrast to *mathematical drawing*, since this has to emphasise accurate, quantitatively correct determination of all details. And of course both tendencies can easily fall into sharpest conflict when one or the other is

[153] Paris 1894. [Related new investigations can b found in the publication: O. Kroh, Eine einzigartige Begabung und deren psychologische Analyse, Göttingen 1922. It originated from the extraordinary calculating achievements oft he mathematician G. Rückle.]

handled too one-sidedly. For example, when one constructs in descriptive geometry many individual points of a curve with great difficulty, but due to the lack of the necessary drawing skills these points might be quite inaccurate and the drawing person might not have the correct idea of the curve's shape; he might lie through the points – instead of a regular curve – an impossible scrawl, which in any case does not give a representation of the really spatial relationships to be displayed. Likewise, on the other hand, also the artistic drawing can become a caricature; the details are so blurry that you might think from a distance to have seen something, but from nearby one sees only a indefinable blob. But to my mind, by operating reasonably, both directions could quite well come to terms and complement each other, which would be highly desirable in the interest of the matter. It would not be quite convenient for mathematics, to position itself hostile towards a new, rapidly emerging movement. Some stimulating material in the direction of an agreement **[257]** is contained in the publication by Friedrich Schilling "*Über die Anwendungen der darstellenden Geometrie*",[154] where he deals among others also of the relationships to art.

Schopenhauer's Criticism of Mathematics; Excursus on the Proofs of the Pythagorean Theorem

I like to mention in this connection the often quoted, very sharp criticism of mathematics by the famous philosopher *Arthur Schopenhauer*, because it is quite characteristic of the hostility of more artistically inclined persons against our science. Schopenhauer considers the succession of individual logical conclusions, which a rigorous mathematical proof must contain, as insufficient and unbearable. He wants to become, to a certain extent, be immediately *intuitively* convinced at a glance of the truth of the theorem; thus, he formed for himself the theory that it exists, besides to the deductions emanating from those logical antecedents, yet *another* mathematical method of proof, which directly takes the mathematical truth out of intuition. From this standpoint, he condemned in his major work "Die Welt als Wille und Vorstellung"[155] as well as elsewhere the entire Euclidean system principally and vehemently and especially Euclid's proof of the Pythagorean theorem was an object of his attacks. He calls this "mousetrap proof", that is, which one finally constraints for admitting the correctness of the claim – by insidiously blocking all possible loopholes, one after the other – but never leading to *inner knowledge* of the truth. No mathematician will agree Schopenhauer on these statements; even if one might ascribe to intuition for mathematics an ever high important role as a heuristic principle, promoting science, finally, the last but alone decisive instance

[154] *About the applications of descriptive geometry*. Leipzig and Berlin 1904. Heft 3 of Felix Klein and Eduard Riecke, Neue Beiträge zur Frage des mathematischen und naturwissenchaftlichen Unterrichts an höheren Schulen. Leipzig and Berlin 1904.

[155] The World as Will and Representation. See: Schopenhauser, *Werke* (edited by Frauenstädt; Leipzig 1859). II, pp. 82 sqq. and III, p. 142; also I, p. 135.

is the logical proof emanating from the premises. By the way, I like to refer to the very interestingly written academic *Festrede* "Über Wert und angeblichen Unwert der Mathemtik" by Alfred Pringsheim,[156] in which he exactly discusses extensively Schopenhauer's attacks.

Admittedly, one might agree entirely with Schopenhauer if he would only attack the disrupted, choppy representation form in Euclid's Elements and propose instead a clearer elaboration of the ideas of any proof step and in general, *besides* logic,
[258] a closer consideration of intuition. But even then he would have chosen a not very suited object for his attacks with the Euclidean proof of the Pythagorean theorem; since for me exactly this proof is, according to its idea – if one disregards some external aspects of the Euclidean manner – for *particularly intuitive*. How should be made apparent by the following presentation:

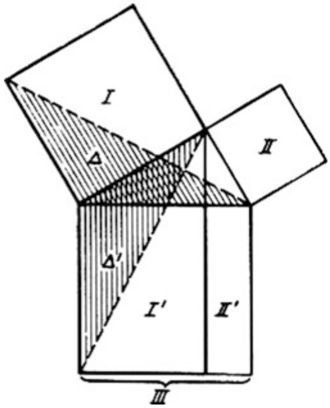

Figure 147

We draw now the figure known to us (Fig. 147) of the right triangle with the squares *I*, *II* on the legs and the square *III* on the hypotenuse; we draw the height of the triangle on the hypotenuse, its prolongation divides the square *III* into the 2 rectangles I' and II'; thus one gets:

$$III = I' + II' . \tag{1}$$

We show now that the rectangle I' is equal to the leg square *1*. To do this, we draw the two obliquely dashed auxiliary straight lines and look at the cross-hatched triangle $\varDelta$ and the vertically hatched $\varDelta'$. The former $\varDelta$ has evidently base and height in common with the square *1*, and has therefore half its size:

$$\varDelta = \frac{1}{2} I$$

[156] About value and alleged worthlessness of Mathematics, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 13, p. 357. München 1904.

likewise, the vertically hatched triangle Δ' is equal to half the rectangle I':

$$\Delta' = \frac{1}{2} I' .$$

Eventually, one sees that both triangles are congruent and therefore also equal:

$$\Delta = \Delta'$$

and therefore, in fact: $I = I'$.

And one can prove that:

$$II = II' ,$$

and considering the fact (1), the Pythagorean theorem follows:

$$III = I + II .$$

Here, then, the proof is realised quite shortly in a manner convincing everybody immediately – as one would think; there, intuition and logic are as intimately connected – and that seems to me the ideal – that *each logical step is immediately brought to intuitive evidence*. Also the lemma $\Delta = \frac{1}{2} I$, which is used here, can be made intuitively entirely clear from Fig. 148, as is well-known, in which Δ arises **[259]** from the half of the square I by translation of the individual horizontal stripes, (Cavalieri's principle!).

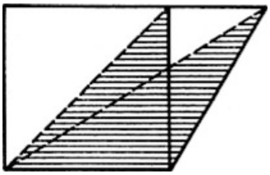

Figure 148

An exposition differing from Euclid's rigid schema and being more flowing and an adapted notation would contribute decisively in order that these simple ideas succeed in emerging correctly and clearly.

I would particularly advocate that one applies more generally in the classroom different shadings, or even better: *different colours* – which is not possible in the present book – to distinguish lines and surfaces, instead of the Euclidean type to mark only the vertices with letters; dealing with a "red" or "yellow" triangle is much more palpable than when one has to identify the vertices E, K, L within a complicated figure.

Therefore, Schopenhauer's attacks against the Euclidean proof are factually entirely unjustified, and this becomes even more clearly when one remarks by what he wants to substitute it. He gives it only for the special case of a right angle and isosceles triangle (see Fig. 149), namely the known Platonic proof; one grasps this

proof in fact at a glance; Schopenhauer restricts himself to demand a similar approach for the general case. But this provides already the Euclidean proof, if done in a reasonable presentation. In fact, essentially, both proofs rely quite evenly on logic and on intuition; only that Schopenhauer's case as the more specific one allows a somewhat simpler settlement, and consequently it is easier then for an untrained person to grasp intuitively the chain of logical conclusions contained in the proof at one stroke.

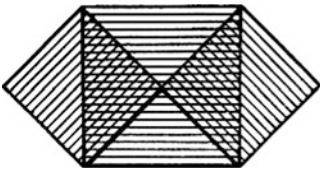

Figure 149

New Impacts from Higher Education

But that's enough of Schopenhauer; let us now finish our comments on the development of geometry teaching in Germany. We had so far basically still followed the line of evolution, originating from those Pestalozzi-Herbartian tendencies for primary school teaching. Now let us see, which suggestions have influenced, with us in Germany, school teaching from mathematics higher education. There we remark a much less satisfactory situation than in other countries. Especially in geometry, the so often lamented phenomenon that university and "secondary" school act
[260] along quite separated paths – without exercising mutual productive interactions. Exceptions were the first half of the 19th century the representatives of the so-called modern geometry, especially Möbius and Steiner, whose works I have quoted in this lecture course many times. Later, however, due to the strong upraise of mathematical science that estrangement increased ever more; only in the last decade, we can happily enough remark again lively efforts to bridge the abyss. As most outstanding evidence for this tendency, I should like to mention again the *Enzyklopädie der Elementarmathematik* by *Heinrich Weber* and *Josef Wellstein*: of that, the most relevant for us are: Volume II [*Elemente der Geometrie*][157] and Volume III [Angewandte Elementarmathematik][158]; in Volume II, you will find: the foundations of geometry (Wellstein), Trigonometry (Weber and Walter Jacobsthal), analytic geometry (Weber), in Volume III vector theory and graphics (Wellstein). However, in this encyclopaedia it is not quite realized what I am wishing for school teaching, as I explained earlier on;[159] especially in the geometrical parts, the authors restrict

[157] Elements of Geometry. [3rd edition, Leipzig 1915.]
[158] Applied Elementary Mathematics. First edition Leipzig 1907. [3rd edition, in two parts, Leipzig 1924.]
[159] See vol. I, p. [4].

themselves frequently on certain subjects, about which they have researched in particular, certainly in very interesting, but quite abstract form – instead of presenting a general orientation about the *overall body* of geometry, as far as this is relevant for school teaching. In contrast, you know what I have repeatedly named as the target of my own lecture course. I aimed to establish an *overall framework* of geometry, in which *all* parts find a convenient place, and which allows an overview over all of them and of their mutual relations. Of course, I could state it only as a postulate that one should try to examine – according to the individual established general aspects, what all of this subject matter is adapted for the school and how far the school system is able to cope with our results.

The Austrian Curriculum of 1900 and the Textbooks by Henrici and Treutlein

Of course, this problem has already been often attacked, but indeed never been resolved so far; thus, I will not omit to mention at least two interesting books that have investigated a large range of the here relevant questions in accordance with consistent criteria. One is the Austrian curriculum of 1900,[160] which adheres to the principles of the Exner-Bonitz reform of 1850. As in 1850, an *Unterstufe* and an *Oberstufe* of the Gymnasium is distinguished (each with 4 years); and in the former, geometry is taught exclusively by intuition with very much teaching of drawing. **[261]** This is continued also in the *Oberstufe*, besides emphasis on logic beginning at that stage. The most interesting point about the curriculum are the *detailed explanations* for mathematics teaching, which reveal a superbly knowledgeable author; I was not able to find out who was the author. We have here an enjoyable contrast to other official curricula, which are usually formulated in the mathematical part so succinctly that one can hardly gather something specific from them.

The second book that I want to mention is the *Lehrbuch der Elementargeometrie* by *Julius Henrici* and *Peter Treutlein*.[161] Here the authors strove successfully to allow for the results of the at that time recent research, *projective geometry*, as well as the *applications*, and also *analytic geometry*, are exposed in organic connection with other subjects, including *trigonometry*. In particular, I mention that the *arrangement of the subject matter occurs according to the classes of geometric transformations*, as we did it earlier on, and as tit was realised he first in Möbius's barycentric calculus: congruence, similarity, perspectivity. Regarding applications, I refer to the fact that at the end of the second part there is a *surveying map* of the Grand Duchy of Baden is (the authors are from Baden), so that the students get a vivid impression of the purpose of trigonometry. To my mind, teaching gains quite extraordinarily by such a vivid reference to local history and geography, which is

[160] Lehrplan und Instruktionen für den Unterricht an Gymnasien in Österreich. Second edition, Vienna 1900.

[161] Treatise of elementary geometry. In 3 parts. Leipzig 1882/83. Many re-editions.

effectively supported by actual carrying out of surveys in the field. Thus, analogously, one should present in mathematics teaching in our [Prussian] schools the Gaussian survey of the Kingdom of Hanover, each student could then learn what it the meaning of the famous triangle Hoher Hagen-Brocken-Inselsberg .

Henrici-Treutlein is therefore an *extremely noteworthy* book. From today's point of view, one might of course regret that the general affinities are missing, thus going beyond the linear transformations of projective geometry, as we investigated them earlier on, and that in connection with that also the modern demands of *functional thinking,* etc. are not considered; a philosophical conclusion is lacking (i.e. a discussion of axioms and the like), as it is now often desired for the upper grades of secondary schools.

We achieved now, gentlemen, the end of our joint observations; although I was able to report to you in the last section already much how now everywhere new life stirs at the schools, I am convinced that the problem of reorganisation of mathemat-
[262] ics teaching, and especially of geometry teaching will become the focus of public interest in the coming years in an even higher degree. All of you, gentlemen, are called to participate in the solution of this important task, by your forces – participating based on independent reflection on all relevant issues and free of the pressure of a predominant, rigid tradition. You will be able to do this when you a sufficient overview both over all relevant areas of science and of the historical development, and for that – I hope – my lecture course should have given you a basis.

Appendix I: Complementary Remarks on Some Issues of Elementary Geometry [263]

1. Reports in the *Enzyklopädie*

If one looks in the scientific part of this book for any issue, which should merit some complementary observations, it is most convenient to first consult Volume III of the *Enzyclopädie der mathematischen Wissenschaften*. We think in particular of the following three reports:

Julius Sommer: Elementare Geometrie vom Standpunkte der neueren Analysis aus. III. A. B. 8 (finished 1914).

Max Zacharias: Elementargeometrie und elementare nichteuklidische Geometrie in synthetischer Behandlung. III. A. B. 9 (finished 1913).

Gustav Berkhan und Wilhelm Franz Meyer: Neuere Dreiecksgeometrie. III. A. B. 10 (finished 1914).

Of these three reports that of Zacharias covers mainly traditional elementary geometry; the other two also cover the elementary configurations, but only in so far as they are treated with the newer methods. Of course, this tendency also prevails in Zacharias' report since he treats the axiomatic issues extensively and non-Euclidean geometry, but not as one-sidedly as do the others. Sommer's report studies – as the title already suggests – a number of questions that have already been discussed in the first volume of the present work. For example, he reports about the feasibility of geometric constructions, on rotation groups of regular polyhedra and on those developments in spherical trigonometry, which are linked with the names of Gauß, Möbius, Klein and Study. The report by Berkhan-Meyer is dedicated to the teaching of the remarkable points, straight lines, circles, and conic sections of the triangle and aims especially to show, by means of the concept of transformation, the relationships that exist between many seemingly isolated and side by side theorems of that kind.

2. The Classification of Geometrical Construction Tasks [264]

Any geometric task can be formulated, if we restrict ourselves to the case of the plane, as follows: Given a figure F of points, straight lines, circles and other curves of the plane E. In E we search for a figure F^* whose points, straight lines, etc. are

F. Klein, *Elementary Mathematics from a Higher Standpoint*,
DOI 10.1007/978-3-662-49445-5

in a prescribed relation to those of F. According to tradition and not due to sound didactical reasons, one admits in elementary mathematics as curves almost only circles, or, at best, the remaining curves of the second order. Since such a curve is determined by 5 points, a circle by 3 and a straight line precisely by 2 points, then every figure F, in so far as only a finite number of these configurations is essential for it, is determined by a finite number of points. The same applies to F^*. The task is then: the points of F^* should be determined by those of F. Given the constraints mentioned, the analytical approach leads to a finite number of equations between the coordinates of the points searched after and those of the given points. If at least one of these equations is transcendent, then the task itself is called transcendent, otherwise it is called algebraic. Furthermore, an algebraic task is called linear or of the first degree, if the calculation of the unknown coordinates leads to equations of only the first degree; however, when quadratic equations occur or such equations of higher degree, which can be replaced by a succession of quadratic equations, it is called quadratic or of second degree. A corresponding meaning is expressed by the statement that an object is of the n-th degree. In addition to this classification there is a second one, which goes back to Möbius,[162] according to which geometric tasks can be distinguished as *projective*, *affine* or *metric* tasks. A task is called projective, if it can be analytically transformed into a system of equations, which is invariant under the group of projective transformations. With an analogous meaning, we have the designations "affine task" and "metric task". However, it is not necessary to establish the system of equations of the task in order to recognise to which group it belongs; this is already announced by its wording. In projective tasks, only projective properties are mentioned and not affine ones, like parallelism, or metric properties, such as the length of a segment or the size of an angle. Of course, besides the already mentioned groups, even others are relevant for tasks of elementary geometry. For example, the famous task of *Apollonius* to construct those circles, which touch three given circles in a plane, belong to the *group of transformations by reciprocal radii* provided we restrict ourselves to point transformations. **[265]** For both the property "circle-to-be" and that of the contact between two circles is invariant with respect to this group. If we connect the just described classification types together, then we have to classify the linear tasks in projective, affine and metric ones and likewise the quadratic tasks and the cubic tasks, etc. Thus it is clear what it means that a task can be projectively linear and another being projectively quadratic.

3. On the Range of Construction of the Most Common Drawing Instruments

The most frequently named mechanical means for solving construction tasks are the following:

[162] Möbius, Baryzentrischer Kalkül, § 139 sqq.

a) the finitely long ruler with one edge and without scale;

b) the parallel ruler (ruler with two parallel edges, without scale);

c) the route exchanger [Streckenübertrager] (a ruler with scale or two sliding marks);

d) the adjustable right angle;

e) the compass.

A task in the theory of constructions is to determine the scope of the individual instruments or of the simultaneous use of several of them. Information on studies about these issues can be found in the report by Sommer or in the earlier mentioned works (see p. [229]) by Federigo Enriques and Adler. With special emphasis I am referring in this connection to the book by Theodor Vahlen: *Konstruktionen und Approximationen*.[163] The book treats, besides the linear, quadratic and cubic constructions, also higher algebraic and transcendental tasks and finally in great detail the approximate constructions. Even there where transcendent cases are involved, the author restricts himself throughout to the use of elementary methods. It is precisely his intention to show their great applicability. The book by Vahlen with its amazingly rich content might represent for every mathematics teacher a treasure trove of suggestions.

I should inform you of some results from the theory of the most used instruments, namely the ruler, the compass and the right angle. Using the ruler only, all projective tasks of first degree can be solved, and only these. If one wants to solve also the projective tasks of second degree by means of the ruler, one has to accept a prescribed conic section K_0 as a further means of construction. A quadratic construction like that to determine the points of intersection of a straight line g and a conic section K would then proceed as follows: by means of a projective transformation K, g is transferred in $K' = K_0$, to g'. One achieves this by the use of **[266]**
the ruler alone. The straight line g' should cut K_0 in the points A_0, B_0; they are the image points of the intersection points A, B, sought which one can now in turn find with the help of the ruler. All linear and quadratic affine tasks are solvable with the ruler alone if one additionally knows the centre of K_0 – and the metric tasks of first and second degree when also the main axes are plotted. The indication of the centre distinguishes the infinitely distant straight line among all the other straight lines of the plane, and by indicating the principal axes the two imaginary circular points are given. Therefore all constructions in which one has to take account of the invariance of these configurations – and these are, at one time, the affine constructions and, at the other time, the metric ones – can be effected. Since as a conic section K_0 can be specially chosen as a circle, it is in agreement with the preceding statements that ruler and compass are sufficient for all tasks of first and second degree. But it also comes as Luigi Mascheroni was the first to have shown, in his *Geometria del compasso*,[164] one will succeed with the compass alone. A proof of this, which reveals the deeper reason of this fact, was given by Adler and is essentially based on the proof that each transformation by reciprocal radii can be realised with the

[163] Leipzig 1911.

[164] Pavia 1797. German edition by J. P. Grüson, Berlin 1825.

compass alone. Each figure F, for whose construction one uses straight lines and circles, can by means of such a transformation be replaced by a figure F' which requires only circles for being drawn. By virtue of the inverse transformation, F' can be transferred in F with the compass alone. Finally, one can, as explained in the above-cited works, replace the compass by the adjustable right angle. The theory of linear and quadratic constructions therefore provides no reason to prefer the compass to the right angle, in the sense that only solutions realised with its aid alone are called rigorous; solely the consideration arising from practical regards, namely that one can work more exactly, in a mechanical sense, with the compass than with the right angle could be invoked for its privileged position.

For the cubic and biquadratic tasks, however, we have noted a great superiority of the right angle with respect to the compass. While these tasks are not solvable with the compass alone or with a simultaneous use of various compasses are, it is possible to solve them with the help of flexible right angles. We want to show the latter.[165]

Analytically, a biquadratic task leads to one or more algebraic equations of fourth
[267] degree in one unknown. The calculation of the roots of an equation of the fourth degree can be reduced to the resolution of an equation of the third degree in one unknown. It is sufficient to show that: every algebraic equation of third degree in one unknown can be solved geometrically by adjustable right angles. For this we need only to use *Lill's method for the resolution of algebraic equations*, well-known from the theory of graphical methods. The equation in question should be brought into the form

$$1 \cdot x^3 + a_1 x^2 + a_2 x + a_3 = 0 \,.$$

We choose a unit of length and represent the coefficients a_1, a_2, a_3 as segments. Then we draw (see Fig. 150) the rectangular polygonal path $ABCDE$, where $AB = 1$, $BC = a_1$, $CD = a_2$ and $DE = a_3$. The direction of AB is arbitrary, we choose those of the other segments according to the following rule. The transition from one segment to the other by turning right when its coefficients have the same sign, and by turning left, when the signs are opposite. In BC we carry a segment $FB = x$, which ends in B positively measuring in the sense arising from a right turn of AB and construct the polygonal path $AFGH$, being rectangular in F and G; G will lie on DC, H on DE.

We claim that the measure of HE, taken with a suitable sign, is equal to the value which the function:

$$y = x^3 + a_1 x^2 + a_2 x + a_3$$

assumes for the chosen x. It holds namely, if we always measure positively in the sense arising from AB by successive right turnings:

$$FC = x + a_1 \,,$$
$$GC = x(x + a_1) = x^2 + a_1 x \,,$$

[165] The following is based on the presentation in the quoted book by August Adler, pp. 259 sqq.

since the triangle ABF is similar to the triangle FCG.

$$GD = x^2 + a_1 x + a_2$$
$$HD = x\left(x^2 + a_1 x + a_2\right) = x^3 + a_1 x^2 + a_2 x\,,$$

since the triangles GDH and ABF are similar.

Finally, from $HE = HD - ED = HD + DE$ follows:

$$HE = x^3 + a_1 x^2 + a_2 x + a_3\,.$$

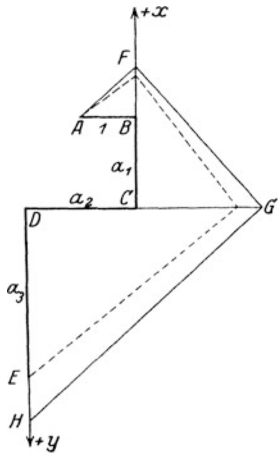

Figure 150

The equation is now solved by constructing such polygonal paths, for which H coincides with E (in Fig. 150 the dotted path) and then determines the measure and the sign of the associated FB. Such "resolving paths" can be found, however, with two adjustable right angles at once. Thus our theorem is proved.[166] **[268]**

Eduard Study supports the requirement of limiting the constructive aids, from geometric considerations, in a paper on the already above-mentioned contact problem of Apollonius (Mathematische Annalen, Vol. 49, 1897). We said earlier that this problem belongs to the group of transformations by reciprocal radii. If we allow only biunique point transformations of space, then G should be the largest group, which transfers tangent circles again into tangent circles. The problem of Apollonius with respect to G is equivalent to the following: Given 3 circles k_1', k_2', k_3' on a sphere K'. We are searching for those circles k' of the sphere, which touch k_1', k_2', k_3'. Because you can always find, for 3 circles k_1, k_2, k_3 in a plane K and the tangent circles k, a transformation belonging to G which transfers the unprimed

[166] One understands easily how to apply Lill's method to algebraic equations of an arbitrarily high degree; see for details *Horst von Sanden*, Praktische Analysis. Second edition, Leipzig 1923.

circles in the primed ones. The plane K corresponds then to the sphere K'. Study's requirement is now to find such a solution of the plane problem, which remains transferable step by step to the spatial problem equivalent to it. Then, of course, only construction instruments may be used, which are invariant with respect to G. Thus, to draw straight lines, the use of ruler is excluded, since the centre of a circle is not associated invariantly with the circle, with respect to G; the centre of the circle is not transferred into that of the transformed circle. Yet, instruments can be used which allow us to lay a circle through three given points A, B, C, since by a transformation such a circle is transferred into the corresponding circle laid through the three points A', B', C'. As such an instrument one can propose an adjustable angle.

First, one has to adjust it thus that its vertex will lie on, say, C, while the legs will pass through A and B. Then C should be moved thus that A and B remain on the legs. Due to the inscribed angle theorem, C would thereby describe the circle sought. This instrument, however, can only be used when also a part of the circle plane is given. For constructions in the space, such as those on the sphere, it is not applicable. This disadvantage does not hold for the bendable circle ruler of E. Tschebyscheff. This consists of a long, elastic steel strip whose back is inserted into a chain of interrelated chain links. If the whole gets bent, the links of the chain form a regular polygon, to which the strip adjoins tangentially. A description of this
[269] ruler, which is originally conceived as a tool for drawing very shallow circle arcs, can be found in a paper by F. Helmert.[167]

We mentioned this instrument to show that especially theoretical requirements can happen to employ entirely different tools than ruler and compass. If we apply Study's demand to the projective, affine and metric group, it results that – with respect to the metric group – ruler, fixed adjusted compass, right angle, even any movable rigid body are invariant drawing tools, while for the affine and projective group among the ones just mentioned it is only the ruler. In any case, never turn down a preferential position of the compass with regard to the right angle. Summing up, we can thus say there is no valid reason to restrict ourselves to ruler and compass in constructions – and therefore to exclude the adjustable right angle. The practice of drawing also provides no reason since this requires just the greatest possible freedom of handling in the use of instruments.

There are many other instruments for the construction of higher curves, which are all correct in theory, but in practice are subject to all sorts of errors. In the theory of graphical methods, alongside the old construction means, an extensive use of transcendental curves and all possible transformations is practiced. Also for the different types of graph papers that are used for this, the construction area should be indicated. Regarding the scope of nomographic methods, one should read the summary report by *Paul Luckey*: "Die Verstreckung (Anamorphose) und die nomographische Ordnung" in volume 4 (1924) of the *Zeitschrift für angewandte Mathematik und Mechanik*.

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[167] *Zeitschrift für Vermessungswesen*, vol. VI, 1877, pp. 147 sqq. – Whether the circle ruler is applicable *in pratice* for constructions on the sphere should be left undecided.

Basically not a mathematical instrument is the so-called spline of the technicians, i.e. a flat sheet encircled by a pleasant perimeter, of which the designer always uses that part of the contour, which seems to represent the most pleasing solution for his purposes.

4. On the Application of Transformations to Simplify Geometrical Tasks

An often successful method for solving geometric tasks consists in transferring the given task in an easier one by applying a suitable transformation and to return, after its solution, via the inverse transformation. Of course, the transformation must leave unchanged the properties of the figure essential for the task. In the representations destined for teaching at schools one finds used, besides the main group, the transformation by reciprocal radii. A very nice collection of examples of this kind is published in a booklet by *Bruno Kerst* in the collection *Mathematisch-physikalische Bibliothek*.[168] The affine and projective group, however, one will find preferably used in the books on descriptive geometry. One in the classroom neglected or at least not clearly presented transformation is the *dilatation*. It is used in some solutions for the contact problem of *Apollonius* and in the very elementary task to draw the common tangents of two circles. To interpret this task correctly, it is necessary to apply the terms "*oriented circle*" and "*oriented straight line*". The oriented circle, also called *cycle*, is a circle with a given direction of travelling round its circumference. Its radius should be considered as positive when we travel counterclockwise, otherwise negative. Each circle is the support of two cycles. The oriented straight line, also called a spear, is the unlimited straight line with a direction to be traversed. Each straight line is the support of two spears. One can understand the oriented circle as enwrapped by a family of spears. Two circles having the same direction of orientation, of which neither one lies inside the other, have only two spears as external tangents in common (see Fig. 151). Two oppositely oriented circles that do not intersect and where none is enclosing the other, have two spears as inner tangents in common (see. Fig. 152). **[270]**

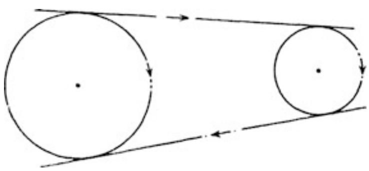

Figure 151

[168] *Bruno Kerst*, Methoden zur Lösung geometrischer Probleme. Leipzig 1916.

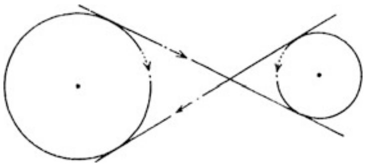

Figure 152

When one looks at the above figures, you will think immediately of driving wheels connected by belts, which in fact circulate in the same direction when the belts represent external tangents, but in opposite directions circulate if they are crossed over and thus form internal tangents. By the normal direction of a spear, we now want to understand the direction of traversing of a particular spear, and into which direction that spear is transferred, if one lets it rotate counterclockwise around any of its points by $\frac{\pi}{2}$. We define now the dilatation as a contact trans-
[271] formation by which all spears are translated in the normal direction by the same amount a paralleled to it. It is evident that this transformation transfers an oriented circle of radius r into a *concentric* circle of radius $r - a$; the new circle will have the same or opposite direction of rotation, depending on $r - a$ and r having equal or opposite sign. The two cycles, which decompose into a circle will disentangle; one will shrink, the other will expand. If the radius of a cycle will increase by the amount a of the dilatation, then the radii of the cycles with the same direction with also increase by the same amount a, while the radii of cycles with the opposite direction will diminish by this same amount. Furthermore, the tangency between two cycles or between one cycle and one spear will be maintained when the line element common to the two configurations is traversed in the same direction. That the dilatations form a group, is immediately clear. – Solving the problem, to find the common external tangents of two circles with radii r_1 and r_2, will be effected, according to these preparations, as follows: (see Fig. 153.)

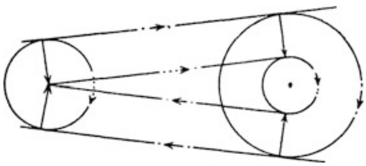

Figure 153

We provide the two circles with a positive sense of rotation, subject them to a dilatation, which has the amount of the smaller radius r_1 and lets shrink the two circles. The smaller of the two circles transfers in its centre, the larger one – while maintaining its centre – in a circle of radius $r_2 - r_1$. Thus, the original task is reduced to the easier one to draw from one point the tangent to a circle. If this is resolved then the dilatation inverse to the first one will lead to the goal. If the inner tangents of two circles have to be constructed, we have to provide them with an opposite

direction of rotation and to subject them to that dilatation by which the smaller circle is transferred in its centre. The greater circle will thereby be transferred into a circle with the radius $r_1 + r_2$ if r_1, r_2 are the absolute values of the radii (see Fig. 154).

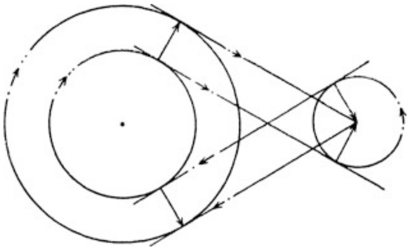

Figure 154

Also the figure of Apollonius problem belongs to the group of dilatations when one replaces the circles occurring there by their cycles. This does not contradict our earlier remark, that the group of transformations by reciprocal radii is the greatest, which transfers tangent circles into circles of this kind. We had restricted, namely, **[272]** that statement to biunique point transformations. However, the dilatation is not a biunique point transformation since for it a point corresponds to a circle; rather, it is a biunique transformation of the directed line elements. By virtue of a suitable dilatation one can replace the Apollonius problem by the simpler one to find all the circles that are tangent to two given circles and go through a given point.

5. New Publications on the Realisation of the *Erlanger Programm*

From the standpoint of the theory of transformation groups, what is commonly counted among elementary geometry, constitutes a colourful mix of components of very different geometries. Apart from the main group there occur, for example, the groups of affine and projective transformations, the transformations by reciprocal radii and the dilatations. Moreover, subgroups of the aforementioned groups occur there sometimes very prominently, in addition to the group of parallel translations and rotations about a point, for example that of area-preserving affinities which characterise the doctrine of the area equality of plane polygons with their propositions – like those about the complementary rhomboids and the area equality of triangles with the same base and height. A requirement of the *Erlanger Programm* is to separate those distinct parts of elementary geometry from each other and to develop them independently. Restricting already to affinities Möbius describes this goal occasionally,[169] after having mentioned some basic propositions of affine geometry, in the following clear terms: “It will therefore probably not

[169] *Möbius*, Gesammelte Werke,, vol. I, pp. 392–393.

be called inconvenient when one tries – starting from these simple propositions as *quasi principles* – to develop those general characteristics which occur proved in geometric publications in a large number, but mixed with other more specific properties and often used with strange means, like trigonometric formulas and the like – if possible to organise them systematically and thus to establish a proper geometric building without *angle measure* and the *Magister Matheseos*[170]." An independent development of affine geometry in this sense can be found in the textbook of analytic geometry by *Lothar Heffter* and *Carl Köhler*.[171] By Wilhelm Blaschke and others was created an affine differential geometry in recent times, namely predom-
[273] inantly as area preserving. A coherent account of these studies is contained in the second volume of Blaschke's differential geometry.[172]

The task to arrange the "inhomogeneous" mass of elementary geometry according to the aspects of the *Erlanger Programm* is what is carried out in the book "*Koordinatengeometrie*" of *Hans Beck*.[173] We want to recommend especially this valuable and insightful book that arose in connection with research by Study and his disciples. To discuss recent generalisations of the whole approach given in the *Erlanger Programm*, is beyond the limits of the conceptions treated in this book.

6. On Descriptive Geometry

There are a great number of new works on descriptive geometry. Among them, the books by *Emil Müller*[174] and *Georg Scheffers*[175] should be especially highlighted. The value of descriptive geometry for technology and in pedagogical terms for the development of geometric intuition should be undeniable. But many mathematicians think of it as a science that is solidified; they see it as a discipline that has stopped to ask the researcher problems, that has reached the end of its development. It could seem there was a certain time that this view was correct; however, today, mainly thanks to the work of Italian and Austrian geometers, it must be contested. In Austria it is Emil Müller, just mentioned, teaching at the Vienna Technical College, who, supported by a large number of disciples, opened new paths in descriptive geometry. A detailed report of this is given by *Erwin Kruppa* in the fourth volume (1924) of the *Zeitschrift für angewandte Mathematik und Mechanik*. To analyse the methods of descriptive geometry of the highest possible geometric standpoint and to reveal its most general principles according to which they can be classified, is the

[170] Möbius means the Pythagorean theorem.
[171] *Lothar Heffter & Carl Köhler*: Lehrbuch der analytischen Geometrie: Grundlagen; Projektive, Euklidische, Nichteuklidische Geometrie. Vol. I Leipzig 1905; vol. II 1923.
[172] *Wilhelm Blaschke*: Vorlesungen über Differentialgeometrie. Vol. I, second edition, Berlin 1924; vol. II 1923.
[173] *Hans Beck*: Koordinaten-Geometrie. Berlin 1919.
[174] Emil Müller: Lehrbuch der darstellenden Geometrie. Two Volumes. Second and third edition, Leipzig 1920.
[175] Georg Scheffers: Lehrbuch der darstellenden Geometrie. Two volumes. Berlin 1919 and 1920.

goal of the work jointly published by Emil Müller and Erwin Kruppa: *Die linearen Abbildungen*.[176]

7. Napier's Rule and the Pentagramma Mirificum

Napier's rule serves, as is well-known, to calculate rectangular spherical triangles of Euler type (i.e. spherical triangles in elementary understanding; see Vol. 1, p. [189]). It consists in: thinking to have written down the five parts of a rectangular spherical triangle with angles different from $\frac{\pi}{2}$, in a cyclical order corresponding to their natural position, while the legs have been replaced by their complements (see Fig. 155). Then the cosine of any part is, firstly, equal to the product of the sine of the parts separated from it and, secondly, equal to the product of the cotangents of the adjacent parts. **[274]**

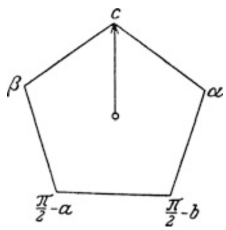

Figure 155

Fig. 155 refers to a triangle *ABC*, whose right angle is *C*. The five parts called *circular* parts by Napier as $c, \beta, \frac{\pi}{2} - a, \frac{\pi}{2} - b, \alpha$ are written at the vertices of a regular pentagon, in a sequence that results when the triangle is travelled around in the counterclockwise direction. Fig. 155 itself has also to be travelled around counterclockwise; the hypotenuse c is highlighted by a pointer radiating from the centre of the pentagon. In usual teaching, Napier's rule serves merely as a memory aid – it does not even think to ask if it expresses some geometric principles. After the 10 formulas for the right-angled spherical triangle have been derived, the rule is simply learned to facilitate mastery of this group of formulas. This type of treatment passes carelessly by the beautiful and easily understandable considerations by Napier. In his *Mirifici Logarithmorum Canonis Descriptio* of 1619 (Lib. II, Chap. IV, pp. 30 sqq.) Napier deduces the rule from the following figure. *ABC* should be again a rectangular spherical triangle at *C* (see Fig. 156, which is intended as a stereographic projection). The great circles, on which the two legs *BC* and *CA* are lying, are denoted by k_1 and k_2, the great circle of the hypotenuse *AB* for a reason, which will become immediately apparent, with k_4.

We construct now those two great circles for which the endpoints of the hypotenuse are poles; the great circle belonging to *A* should be denoted by k_3, the one

[176] Leipzig and Vienna 1923.

belonging to B with k_5. The following will intersect at right angles: k_1 and k_2 in C, k_2 and k_3 in F, k_3 and k_4 at K, k_4 and k_5 in D, k_5 and k_1 in H. This results in a closed chain of five rectangular triangles whose right angles lie at C, D, F, H, K and their hypotenuses are forming the pentagon $AEGJB$. This pentagon is called *mirificum*
[275] because of its remarkable properties as Pentagramma; in particular it has attracted various times the interest of *Gauß*.[177]

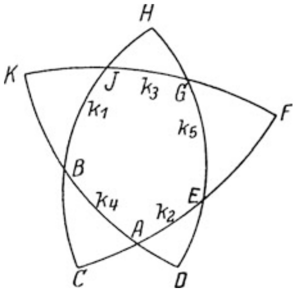

Figure 156

It is easily seen that the vertices of the pentagon are all poles of the great circles $k_1, \ldots, k_5$ are, namely already due to the construction: A is pole of k_3, B is pole of k_5. Moreover, E as the intersection of the two circles k_5 and k_2, both perpendicular to k_1, is pole of k_1 and correspondingly G is pole of k_4, J is pole of k_2. It follows that the sides $BD, CE, AF, DG, EH, FJ, GK, HB, JC, KA$ are throughout equal to $\frac{\pi}{2}$. Accordingly, the triangle ADE can be constructed from ABC so that it prolongs the hypotenuse BA and the leg CA by their complements. By successive application of this construction procedure the original triangle transforms into the full chain of triangles.

The cyclic arrangement of circular parts of the triangle ABC may lead again to Fig. 155. We calculate now the circular parts of ADE from those of ABC. It is:

$$\frac{\pi}{2} - AD = \frac{\pi}{2} - \left(\frac{\pi}{2} - c\right) = c\,; \quad \angle DAE = a, AE = \tfrac{\pi}{2} - b$$

and (think ED as extended from D to the intersection with BC, extended from C):

If we write the circular parts of the triangle ADE in the same way as before for ABC, we obtain again exactly the Fig. 155, except the position of the hypotenuse pointer. The sequence and size of the circular parts will not have changed, only its meaning has become another one. If we remember that each of the triangles following ADE is in the same relationship to the preceding one as ADE to ABC; thus we can pronounce the proposition:

Fig. 155 is, except for the position of hypotenuse pointer, invariant with respect to the group of operations, which replaces any triangle of the chain drawn in Fig. 156 by another triangle also belonging to the chain.

[177] See in Gauß, *Gesammelte Werke*, vol. VIII, pp. 112 sqq., Göttingen 1900, the remarks by Fricke on the eleven pentagramma fragments.

That this simple relationship results, we owe much to the fact that *among the circular parts are counted the complements of the legs and not the legs themselves*.[178] It has to be expressly noted that all triangles must be travelled around in the same direction. Finally, observing the importance of the parts, we note that the hypotenuse pointer of Figure 155 performs, in the transition from *ABC* to *ADE*, a positive rotation by $3 \cdot \frac{2\pi}{5}$ (see Fig. 157), and the same applies to the transition from *ADE* to *EFG*. Therefore, when we traverse the triangle chain, each of the circular parts of *ABC* will become once a hypotenuse, twice a leg complement and twice an acute angle. The five sides of the *Pentagramma mirificum* will be constituted by the five circular parts, however, in a modified order. If we have derived for *ABC* the formula **[276]**

$$\cos c = \sin\left(\frac{\pi}{2} - a\right) \cdot \sin\left(\frac{\pi}{2} - b\right)$$

then there are *five formulas comprised in one,* since the properties to be a hypotenuse or a leg proved to be accidental and that only the relations of arrangement are essential. The same applies to the formula

$$\cos c = \cot\alpha \cdot \cot\beta\,,$$

which results from some of the previous five by elimination. If we consider in Fig. 155 the letters as fixed, but the pentagon as rotatable around its centre and rigidly connected with the hypotenuse pointer, then we can consider the group of operations that transform a triangle of our chain into another one, under the image of the group of rotations which transfer the pentagon into itself.

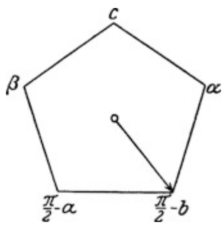

Figure 157

We have not followed Napier's presentation strictly. However, one point in Napier's versions seems particularly noteworthy and to be little known; therefore it should not be overlooked by us. We mean the way how he located his figure in the sky. He goes outside for this purpose, he started from the triangle constituted by the pole, North Point and setting sun, which has a right angle at the North Point. If we assume, as in Fig. 156, *B* as a pole, *C* as North Point, *A* as a place of the setting sun, he had therefore as the great circle k_1 the local meridian, as k_2 the horizon, as k_4 the meridian of the sun, as k_5 the celestial equator, and as k_3 the great circle,

[178] Like Napier, one can also choose the legs and the complements of the hypotenuse and of its adjacent angles as circular parts.

which has the sun as its pole and should be briefly called the accompanying circle to the sun. The *Pentagramma mirificum* then has as vertices the setting sun, West Point, the intersection of the celestial equator with the accompanying circle of the sun, zenith and pole. Our figure would thereby presuppose northern declination of the sun.

(S. [Seyfarth])

Appendix II: Additions About Geometry Teaching in the Individual Countries [277]

The parts on geometry teaching treated in the final chapter were written around the year 1908, just before the work of the *Internationale Mathematische Unterrichtskommission* (IMUK) started. Since then, and around the time that the war began, the actual investigations of IMUK were concluded. It has effected in almost all civilized countries a profound transformation of the organisation of teaching. One could conclude from this that all the presentations published by IMUK would have to be set aside as out-dated. In contrast, we are convinced that in the majority of the IMUK reports the richness of ideas that are of lasting value is far too great to permit that judgment to be allowed, even in the most remote manner.

A presentation of what took place in the organisation of teaching in the post-IMUK time, in particular in foreign countries, reveals something quite difficult. It is not easy to obtain sufficient material for a reliable assessment of the recent movements in the various countries – because, in particular, so much is still in flux. We will therefore limit ourselves to communicating some individual facts, knowledge of which we owe either to IMUK or which we obtained otherwise by chance.

A typical pattern of the recent developments should however be commented already here. It is the low estimation which mathematics and science are given in comparison with the mother language, literature, history and art as educational means and educational elements. One would have expected the exact opposite, given the circumstances of the time: since technology, which is so inseparably linked with those areas of science and the immense importance of which for the life of people – for national defence in the war, for prosperity in peace enabling all cultural work – has never imposed itself so forcefully on the consciousness of people as in our times. But it may be that the enormous extent to which the advance [278]
of technology has taken place and continues unabated, has, considering the psychical strength and the absorbing capacity of most people, meant for most an excess of technology and led to satiety. So the movement hostile to mathematics and the natural sciences, which one finds in all countries involved in the war, can probably be explained, at least in part, as due to fatigue.

In France this sentiment led in 1923 to the reform by the Education Minister Bérard, which declared Latin as a mandatory subject for the first four years of all secondary schools; he thus wanted to eliminate the purely realistic from school. This reform was opposed particularly by *Herriot*, *Painlevé* and *Leygues*. Painlevé

is the famous French mathematician and Leygues the Minister under whose leadership the curricular reform of 1902 had been realised. Herriot, when he became Prime Minister, essentially restored the former situation, which allowed the secondary schools without Latin to be of equal status with other secondary schools. The position of mathematics and especially of science teaching in the Italian schools also seems very sad to us. There you have, according to a plan, which has been decreed by the fascist Minister Gentile, a range of school types in which there is very little mathematics and no science teaching at all. We have reported earlier (see Vol. I, pp. [298] sqq.) about the inadequate value attached to our discipline in the Prussian school reform. Quite a different opinion to that prevailing in the Prussian education reform appears to be held by the Chancellor, *Luther*. For the speech he gave at the opening of the *Deutsches Museum* in Munich contained an unrestricted commitment to the value and dignity of technical work, supported by strong inner conviction.

In Russia the mentioned disciplines experience an extraordinarily high esteem, but only insofar as they are in an obvious way closely related with issues of practical life. Information about Russia and the aforementioned reforms can be found in a publication issued by our Ministry of the Interior under the title, "European education reforms since the World War"[179] About the French educational reform of 1923, which was rescinded, Bérard himself gave a report on the discussions following its proposals in the French Chamber.[180]

[279] The facts relating to the reform of Italy, edited by the Italian Ministry of Education, are collected in a book published in 1924 under the title "Raccolta di Norme e Regolamentari sull'Ordinamento dell'Istruzione Media".[181]

In the following discussions on teaching issues we will restrict ourselves as in the final chapter to England, France, Italy and Germany. We begin with a discussion of English school organisation.

1. England

Concerning the school system in England we make use of the reports of the English Subcommittee of IMUK which have been grouped together in two volumes under the title "The Teaching of Mathematics in the United Kingdom",[182] and also a Ger-

[179] "Europäische Unterrichtsreformen seit dem Weltkrieg", bearbeitet im Reichsministerium des Innern. Leipzig 1924.
[180] Leon Bérard, *Pour la réforme cIassique de l'enseignement secondaire*. Paris 1923.
[181] Roma 1924.
[182] The exact title is: Board of Education – *Special Reports on Educational Subjects*; Volume 26 and 27, *The Teaching of Mathematics in the United Kingdom*. Being a series of Papers prepared for the International Commission on the Teaching of Mathematics, Part I and Part II. London, Published by His Majesty's Stationery Office 1912.

man IMUK report by Georg Wolff.[183] From the latter, we first extract the fact of the great complexity and missing generality in the organisation of the English school system. "It was told to me," Wolff reports – characteristically enough – on p. 24 of his book, "that the inspectors had to make an inspection of a school. Before beginning their inspection, the headteacher and other teachers had to report exactly on the school's structure and the curriculum for the various disciplines. Nevertheless, the inspectors had difficulty in understanding how that school was functioning, and only on the third day of their visit did they begin to form their first ideas of that system." The almost complete independence of each school from a central state authority makes it possible to adapt its organisation to the individual needs of students to a degree unknown to us. Thus, many secondary schools have for the lower grades a common structure, followed by a division into several departments for the upper grades. As such divisions one usually meets a classical side with Latin and Greek, a modern side with French and German and a science side with emphasis on mathematics and the sciences. Special measures are arranged which allow transfers from one division to another. When there exist parallel grades, one often separates the more gifted students from those less gifted; moreover, for the former it is also made possible to proceed faster than the others by promotion to senior classes after six months or after a school term, i.e. a third of a year. Regarding promotion, it is **[280]** not always necessary to move up a grade in all subjects. Thus, a student may study mathematics in a different grade from that in which he is studying languages. Thus, that freedom of movement, which is currently often required for our secondary schools is traditionally practiced in England without restriction. This freedom in the organisation is, however, in contrast to a profound internal bondage regarding the practice of teaching which has its roots in basically two facts; namely in the already described centralisation of the examination system, and in the lack of scientific and educational training of a great percentage of the teachers at the secondary schools in England.

It is largely due to these circumstances that the movement for the reform of mathematics teaching in England makes only very slow progress. Euclid is still by and large the standard text for geometry teaching, Thus stereometry is particularly blatantly affected by Euclid's influence. It is given little consideration by Euclid, and as a consequence it experiences the same step-motherly treatment at the average English school.

Among the Englishmen who played a role in the recent history of mathematics teaching, we have emphasised, on pp. [233]–[234] and [236] *Perry* and *Branford*. Perry's radical program has aroused strong opposition from many of his compatriots, and as we stressed ourselves, to a large extent justifiably. No one today hardly ever thinks seriously of an even moderately complete implementation of his proposals as they concern the secondary schools. However, Perry's approach yielded the very valuable result that a great number of mathematics teachers became aware

[183] G. *Wolff:* Der mathematische Unterricht der höheren Knabenschulen Englands. Berichte und Mitteilungen, veranlaßt durch die Internationale Mathematische Unterrichtskommission. Zweite Folge. II. Leipzig 1915.

of the need to precede deductive geometry teaching by an experimental preparatory course.

According to Wolff, the influence of Branford has been a minor one. Nevertheless, we should like to describe him as a methodologist of the first rank. A textbook resulting from his ideas is here presented in more detail; it is "A School Course of Mathematics" by *David Mair*.[184] Mair proposes a kind of teaching, which seems to correspond with what one demands today in Germany, where one speaks of "*Arbeitsunterricht*".[185] It is characterised by the following features. The starting point of teaching is based on suitably selected problems. Solving these problems should be brought about through discussion between the students and the teacher with a minimum of guidance on the part of the latter. The more the students develop
[281] the solution themselves and without the help of the teacher, the more safely, thus Mair rightly argues, they will have mastered the knowledge acquired, and the more valuable is the training of the mind that they receive during the process. But if one wants the students to be able independently to make mathematical considerations, then they must have a certain stock of concrete mathematical experience. When students are slow in thinking, thus Mair says further, it will be good to increase that stock by drawing and measuring exercises, thus to continue the course mainly experimentally, and to return only after that to reflecting about the results found by experience. "To repeat the words of another's reasoning is not to reason" (p. 11). For more complex problems, it is sometimes recommended to pay particular attention to the speed of progression in the lessons. The most valid teaching method for both the practical and the formal purpose, namely, discovery by the students, is only possible with slow progression – when the progression occurs faster, teaching could only be done dogmatically. Even greater haste would confuse the students' minds and could even become a threat to their health. Furthermore, Mair takes special care not to ask for proofs from the students, the necessity for which they are not able to grasp, because such an approach must destroy in the adolescent mind all sense of the nature and purpose of a mathematical proof. He warned against pointing out logical difficulties to students, which they do not feel themselves.

The correct selection of the problems is especially important for such a kind of teaching. It must – as Mair emphasises – be ensured that this intensive method will not restrict too much the theoretical and practical bearing of the knowledge achieved by this teaching. Mair, who wrote his book for age groups corresponding to our lower and middle levels at the Gymnasium, always starts from such problems of practical life, the right treatment of which has sufficiently great importance for intellectual development. He argued thus (see his Preface): the value of a problem can be judged from two points of view: on the one hand, by the value of the knowledge that provides its solution, and, on the other hand, by the educational value, which may be linked to the establishment of the solution. The value of the knowledge in a problem is all the greater the closer it is connected with human life

[184] Oxford. At the Clarendon Press. 1907.

[185] [Translator's note: This means literally: work teaching; it can be understood as a problem-directed teaching method.]

and its interests, and the less the further it moves away from it. For psychological reasons, teaching has only mentally forming power when it progresses from the concrete to the abstract. Such a teaching method is, however, best possible in connection with issues of concrete human interest. Among the problems, proposed by Mair in his book and whose treatment he details in the book, as resulting from free discussion with his students, some should be mentioned: 1. A boy builds in the **[282]**
garden a hiding place for treasure and covers it so that it is indistinguishable from its surroundings. By which measurements, should he indicate that place so that he can find it again. 2. determine the position of a chair in the hallway. 3. copy an exhibited map. 4. a chapter in which the binomial theorem is developed at the end, which begins with the Morse Code and in connection therewith arise simple questions that belong to combinatorics. 5. The calculation with powers and the slide rule will appear next to the question, how can certain calculations already carried out be abbreviated?

Mair's book is, as we have said, written for lower and intermediate grades; apparently it is very little used in schools. As concerns geometry teaching for the upper grades at English schools, it is very difficult to say anything of general validity. Valuable information and suggestions relating to this teaching can be found in the IMUK reports[186] "The Educational Value of Geometry" and "A School Course in Advanced Geometry", of which the first is from the well-known didact *George Carson*, the second from *Clement V. Durell*. Some main ideas in Durell's proposals, of which we do not know to what extent they are realised in schools in England, are: the introduction of imaginary elements in the geometry; a simple manner of using homogeneous coordinates to show the importance of the improper elements; the exploitation of the following transformations for calculations and intuition:

$$x = cx', \qquad y = y',$$
$$x = \frac{px'}{y' + p}, \quad y = \frac{qy'}{y' + p}.$$

With the first one, one can transform – if the constant c is real and determined in a suited manner – the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ into a circle $x^2 + y^2 = r^2$, and likewise when c is imaginary the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ into such a circle.

An entire series of theorems about the circle can thus be transferred to the ellipse and hyperbola. The second transformation is a very simple case of a projective one and can find a similar use. Durell believes that the treatment of orthogonal and central projection can thereby be greatly simplified by, albeit modestly, admitting analytical calculations and not insisting on the purist position that only purely geometric proofs should be allowed.

[186] Pages 257–273 and 351–364 of the first volume of the British reports quoted two footnotes before.

[283] 2. France

We had already noted the strict centralised organisation as the pattern that decisively distinguishes the English school system from the French one. The structure of today's secondary education in France is of the following nature: After the students have passed two "Classes préparatoires" and two "Classes élémentaires", they enter, about 10–11 years old, into the secondary school. This is divided into 3 cycles, of which the first comprises 4 years, the second two years and the third 1 year. In the first cycle, students have the choice between 2 sections A and B. In A, from the first year Latin is an obligatory discipline and from the fourth year Greek. In B the ancient languages fall away, in favour of greater integration of mathematics, science and drawing. The programmes of the two sections are thus that the students are, at the end of the first cycle, in possession of knowledge which forms a whole and can satisfy them. In the second cycle four sections A, B, C, D can be distinguished, which are characterised by the following keywords: A. Latin-Greek, B. Latin-Modern Languages, C. Latin-Mathematics and Natural Sciences, D. Mathematics-Natural Sciences-Modern languages. At the end of the second cycle, the students acquire by an examination the first part of the *baccalauréat*; the third cycle, consisting of only one grade, prepares for the second part of the *baccalauréat*. In the third cycle, the students can choose between the *Classe de Mathématiques* (8 hours mathematics per week) and the *Classe de philosophie* (1 hour weekly mathematics). The successful completion of the *Classe de Mathématiques* is not sufficient for admission to study at a number of colleges. The *École Polytechnique*, for the training of military engineers, the *École Centrale des Arts et Manufactures*, for the training of civil engineers and the Department for Mathematics and the Sciences at the *École normale supérieure*, which forms the teachers of secondary schools, admit their students only on the basis of an entry examination. The preparation for this examination is done for the *École Polytechnique* at the *Classe de Mathématiques spéciales*; to study in these successfully, it is advisable to study earlier at a *Classe de Mathématiques spéciales préparatoire*, of which various exist mainly in Paris. Before entering the *École Centrale*, one has to be prepared again by studying at so-called *Classes de Centrale*. The teaching subjects of these various *Classes* extend into the theory of differential equations and the curves and surfaces. Weekly tests of students outside normal teaching serve for better memorising of the treated subjects. The entrance exams for the above colleges, especially for the *École Polytechnique*, are very difficult. Only very few students manage to
[284] pass the exam after they have spent only one year at the *Classe de Mathématiques spéciales*. Most people learn there for 2 or even 3 years.

Now in regard to the French curricula for mathematics teaching at secondary schools, this can be read in detail in the IMUK report by Th. Rousseau.[187] Only one issue should be commented upon here, on the order of teaching the subjects,

[187] Commission Internationale de l'enseignement Mathématique. Sous-Commission Française. *Rapports*. Volume II. *Enseignement Secondaire*. Publié sous la direction de M. *Charles Bioche*. Paris 1914. S. 76–117.

because this principally distinguishes the French from the German curricula. While in Germany the mathematical syllabus which a given grade has a study is always new compared to the one taken the previous year, this is not generally the case in France. At the secondary school itself the subject matter, at least in terms of mathematics, is arranged in three cycles. Each subsequent cycle has the task, besides introducing the new areas, of ensuring that the previously treated subjects are again taught, but in another manner, taking into account the greater maturity of the students. One can describe this arrangement as one formed by concentric circles; in terms of methodology, in the first circle intuition prevails, in the following ones deduction is increasingly emphasised. In the much used and beautiful textbook of geometry by Jacques Hadamard,[188] which is written for the *Classe de mathématiques*, one finds the geometry established from the foundations and presenting not only those parts that are new for that *Classe*. The subjects already known by its students are covered rather extremely extensively, but at a higher level than was convenient at an earlier age.

3. The Influence of Méray on Geometry Teaching in France

Méray's book, already described here in detail, has influenced the French geometry teaching considerably. The curricula of 1905, the year of publication of the third edition of Méray's book, contain the passage: "Un appel constant à la notion de mouvement semble devoir faciliter l'enseignement de la géométrie; c'est ainsi que sera le parallélisme lié à la notion expérimentale de translation, que l'étude des droites et plans perpendiculaires résultera de la rotation; l'idée d'égalité sera liée à celle du transport des figures, que l'on précisera en introduisant la notion simple d'orientation".[189] But also the resistance against Méray was strong and was due to several reasons. Regarding style, his text is cumbersome and not distinguished by the clarity and elegance that are otherwise often found in French mathematics textbooks. Moreover, many could not become accustomed to axioms other than those expressly stated in Euclid should figure at the head of geometry. Finally, some who **[285]**
agreed to the basic conception of Méray, that the properties of the movements of geometry should form the basis, could not reconcile this with the idea of fusion between plane and spatial geometry. But most serious was the criticism that Méray put too little emphasis on the number of axioms. At this point *Carlo Bourlet* intervened (see p. 245). He drew on the concepts of group and of transformation and outlined that the basic idea of Méray's theory of translations can be brought to the simple

[188] *Leçons de Géométrie élémentaire*. Vol. I: 8th edition 1924. Vol. II: 4th edition 1921.

[189] A constant appeal to the notion of motion seems to facilitate the teaching of geometry; thus will parallelism be related to the experimental notion of translation, the study of straight lines and perpendicular planes will result from rotation; the idea of equality will be linked to the transportation of the figures, which one will need when introducing the simple notion of orientation.

form: The group of translations is an invariant subgroup[190] of the main group of movements. Due to this conception *Émile Borel* and *Bourlet* wrote textbooks in which Méray's structure was simplified, together with a clearer elaboration of his basic conception. More radical were the suggestions by Rousseau in the aforementioned IMUK report. He wants complete renunciation of Euclid and an unrestricted domination of the transformation idea. For geometry textbooks and mathematical elementary instruction he proposes the following order of teaching issues:

1. The beginning should be formed by notions and theorems that belong to the geometry of the most general unique point transformations, i.e. of analysis situs. Of course it is intended an analysis situs on an experimental basis. According to Rousseau, one should treat here concepts like those of the solid, the surface, the line, inside and outside, the cut and the connection. Nothing should impede, says Rousseau, drawing the attention of students to such problems as that of the bridges and islands, of four colours, and the number of sides of a surface.

2. In second place would come the study of movement in general and of rotations with their applications in particular: straight line, perpendicularity, composition of rotations, plane, circle, symmetry, geometry of the family of rays. Here would belong all the properties which are common both to the non-Euclidean geometries and to those of Euclidean geometry. In this part of geometry, in fact, one does not yet make use of the fact that the group of motions has an invariant subgroup.

[286] 3. A third part would be dedicated to translations and their applications: parallelism, metric relations.

4. In a fourth part one would study other transformation groups, such as those of similarity, of transformations by reciprocal radii, etc.

Principally, the here required arrangement and order is not contrary to pedagogical principles. Each of the 4 geometries mentioned can boast quite simple along with difficult problems and an arrangement in concentric circles, as in the traditional structure of the geometry, is also possible here. The old mode of presenting geometry due to Euclid classifies essentially according to figures (straight line, triangle, square, circle, plane, spatial figures). In many textbooks this older viewpoint is blended with the new ones just described.

One of the oldest *German* textbooks, in which the transformation viewpoint appears is that, written under the influence of Möbius "Lehrgebäude der niederen Geometrie", for use at Gymnasia and secondary modern schools, by *Carl Anton Bretschneider* (Jena, Frommann 1844). In this textbook, the usual division into plane geometry and solid geometry is abandoned and replaced by the following:

1. Synthetic geometry:

a) the geometry of position.

b) the geometry of shape.

[190] The subgroup g of a group G is called invariant, when at a certain process of composition of any transformation T of g with any transformation S of G again a transformation of g results. (The group is invariant regarding the respective process.) That process is defined as follows: when S^{-1} is the inverse of S: form the product $S^* = S^{-1}T$ and then S^*S, which is identical with $S^{-1}TS$. The transformation S must belong to the group g.

c) the geometry of measure.
2. Analytical Geometry:
a) goniometry.
b) trigonometry.
c) coordinate geometry.

Also dominated by the concept of transformation is the textbook of elementary geometry by *Henrici* and *Treutlein*, already mentioned on p. [261].

4. Italy

According to the decree of the former Minister *Giovanni Gentile* of 6 May 1923, the way to higher education in Italy passes either by the three-year *Liceo classico*, or by the four-year *Liceo scientifico*. To enter the *Liceo classico* one has to pass an entrance examination after having undergone 4 years elementary school and 5 years secondary school, called Gymnasium (*ginnasio*). To enter the Liceo scientifico, it is sufficient, after the same four years of elementary school time: to have four years in a Gymnasium, or four years at another middle school. Such middle schools are: the *scuola complementare*, roughly corresponding to the Prussian secondary modern schools, and the lower course of the *Istituto tecnico*, whose objective is to train for medium level technical professions. How strongly mathematics and **[287]** the sciences have been reduced in Italy by Gentile one can see from the following information, which one finds in the publication of the Italian Ministry of Education already mentioned on p. [279]. For the Gymnasium, we proceed from the lower to the upper grades (reckoning is not indicated separately): the hours for mathematics are $1, 2, 2, 2, 2$; for physics, chemistry and biology no teaching is prescribed. In the *Liceo classico* the number of hours, to be shared between mathematics and physics is $4, 4, 5$. The numbers for chemistry and biology are in total $3, 2, 3$. In contrast, we find history and geography together in Gymnasium with $5, 5, 4, 3, 3$, and at the *Liceo classico* for history alone $3, 3, 3$; for philosophy and social studies also $3, 3, 3$ and for history of art $2, 2$. The total number of hours for scientific teaching varies in the Gymnasium between 21 and 24, in the *Liceo classico* between 25 and 26 per week. For the *Liceo scientifico* which should emphasise the mathematical-scientific element especially, the numbers of hours for mathematics and physics are in total $5, 5, 6, 6$, for philosophy and social studies $4, 4$, for history $3, 3, 2, 2$, for biology, chemistry and geography also $3, 3, 2, 2$. The lower course of the *Istituto tecnico* provides for mathematics (reckoning included) $2, 2, 4, 4$ weekly hours, but as in the Gymnasium neither physics nor chemistry nor biology. In the aforementioned *Raccolta*, no curricula are specified. It is only indicated what is demanded in the various examinations.

As a characteristic innovation it should be mentioned that at the *Liceo scientifico*, in the hours for philosophy and social studies, besides the disciplines already defined by this title, the history of mathematics and the sciences have also to be taught; yet, this teaching will generally not be in the hands of a scientist. The issues

to be taught are according to the source (*Raccolta* p. 369) and in free translation: The problem of mathematics and science in its historical development. The science of the ancients (mathematics, physics, chemistry, astronomy). Medieval science. Science in the Renaissance and Naturalism (Telesius, Campanella, Copernicus, Gilbert). The great question of the Ptolemaic and Copernican system (Galileo). The problem of scientific method (Bacon, Descartes). Modern science. New theories about the nature of science (Croce, Maxwell, Mach, Poincaré).

A textbook very helpful for such teaching has recently be published by Gino Loria "Pagine di Storia della Scienza".[191] Of recent Italian textbooks, which are
[288] intended for geometry teaching at secondary schools, we have two to hand. One is due to *Cesare Burali-Forti* and *Roberto Marcolongo* and is intended for the upper course of the *Istituti tecnici*;[192] the other is written by G. Predella for use at the *Licei*.[193] It is of course impossible to derive from the character of these two books typical features of Italian teaching with security; however, we mention them as they present new aspects compared to those previously discussed. The first of them presents geometry with constant use of the vector concept. The chapter headings are: generalities about vectors; sum of two vectors, the product of a vector by a real number; scalar product of two vectors; the rotation, including a paragraph: the operator i (rotation through a right angle); circular functions; plane trigonometry; vector product; spherical trigonometry; conic sections; various considerations (concept of power of a point with respect to a circle transformations by reciprocal radii, etc.). In the preface of the book the author explains how: vectors are now commonly used in university teaching; they apply an algorithm that is similar to the common one in algebra and as simple as that; they are so geometrically suggestive despite their algebraic algorithm – therefore they should not remain unknown to the students of the upper course of the middle schools.

For us, the book appears to inundated by formulas and it is written quite abstractly. This holds in part to an even greater degree for the textbook of geometry by Predella. It begins with a chapter in which the concepts of quantity, upper limit, irrational number are discussed with great rigour, in order for them to be applied to a few theorems in planimetry. In the further course, the book does not rise above the simplest issues of solid geometry, which may well be related to the small number of hours, available for mathematics.

On p. [247] we have called the idea of fusion between solid geometry and plane geometry as being especially influential in the Italian geometry teaching. However, this appreciation is no longer valid under the current circumstances. Even in the deliberations of the IMUK Congress of Milan, in 1911, it became clear that the efforts for fusion had been completely marginalised in Italy. This statement had to be made at a time, when Peter Treutlein tried, by his translation of the fusionist stan-

[191] Published in the Biblioteca Paravia "Storia e Pensiero".

[192] *Corso di Matematica pel Secondo Biennio degli Istituti Tecnici*. Vol. II Geometria. Firenze 1921.

[193] *Geometria ad uso dei licei*. G. B. Paraiva, Torino-Milano.

dard Italian work,[194] the “Elementi di Geometria” by *Giulio Lazzeri* and *Anselmo Bassani*,[195] to arouse German interest in the concept of fusion.

5. Germany (On the Further Development of the Prussian School Reform) [289]

Already in Volume I, we have reported on the history of mathematics teaching in Germany and in particular on the role of mathematics and the sciences in the Prussian educational reform beginning in 1924.[196] One of the central ideas of this reform, as it was originally intended, aimed at establishing four strictly different types of secondary schools, each of which would represent a particular aspect of culture. The *altsprachliches Gymnasium* would focus its teaching on the relation between the German and the ancient cultures. The *Realgymnasium* has as its objective the study of modern European culture and, since the modern languages are its main disciplines, is called *neusprachliches Gymnasium*. Mathematics and the sciences are passed to the secondary modern school, *Oberrealschule*, as its dominant disciplines; in addition to realising purely disciplinary objectives, these schools are also required, in particular, to emphasise the cultural achievements of mathematics and the sciences. The objective of the *deutsche Oberschule*, finally, is to convey an understanding of German culture: German language, history and geography are its main disciplines.

The requirement of “pure school types” impacted upon mathematics and the natural sciences since in the timetables originally proposed by the Prussian Ministry of Education these subjects lost strongly in importance in all schools except the *Oberrealschule*. To appreciate the consequences of this restraint on the training of engineers and physicians and the mathematical-scientific education for the other professions, it is necessary to know that the number of *Oberrealschulen* in Prussia is still quite low, that it is not envisaged to increase their number and that these disciplines had previously played a significant role at the more numerous *Realgymnasien* which they would now lose.

The battle, which various sides began to fight against the Prussian education reform, resulted in a modification of the timetables which brought a reduction of the typification aims. At the *Gymnasium* and *Realgymnasium*, mathematics and science teaching was strengthened somewhat, while at the *Oberrealschule* it was weakened. We cannot discuss all these issues in detail, only the following may be highlighted:

1. At no secondary school type did mathematics and science teaching receive the number of weekly hours it had before the reform. The requirements of the revised *Meran* curricula with regard to the timetables are not realised. [290]

194 Lazzeri und Bassani, *Elemente der Geometrie*, deutsch von Peter Treutlein. Leipzig 1911.
195 First edition Livorno 1891; second edition 1898.
196 See Vol. I, pp. [291] sqq.

2. The scientific subject that is most marginalised by the Prussian reforms is biology. It may be that – regarding recent developments in biology – long outdated ideological reasons were not without influence on the low assessment of this science.

3. If we compare the organisation (not the spirit which dominates teaching and education) of the Prussian secondary education with the English and French ones, one has to note a decreasing regard for the individuality of the student: from England to France to Prussia. In England we find the greatest freedom of organisation for the sake of the student. In France, on three occasions, namely on each new entry into one of the three cycles, there is the possibility of choosing a school type that corresponds best to the giftedness and interests of the student. In Prussia, the student generally, that is when he lives in a medium-sized town with only one type of secondary school for boys, is restricted to the *one-sided* type of school of his home town.

4. Recently, new guidelines for the curricula of the secondary schools of Prussia were published.[197] As imagined in Volume I it was confirmed that the principles of the mathematics education reform would play a full role in them. Apart from a few deviations, the new Prussian curriculum agrees, for mathematics, with the revised *Meran* curricula. Therefore, at all Prussian secondary schools types function will constitute the central concept of mathematics teaching and at all school types the elements of infinitesimal calculus should be taught. The formation of space intuition is brought to the fore, the history of mathematics should in general be considered and applications should be properly emphasised. Geometric drawing should be a substantial component of mathematics teaching; *the entire descriptive geometry is integrated*. However welcome this last requirement, it is nevertheless very doubtful whether its realisation will succeed due to lack of available teaching time.

5. In the methodological conception of the Prussian curriculum, two require-
[291] ments play a particularly prominent role: the *Arbeitsunterricht* and the *concentration*. We have tried to characterise what is meant by *Arbeitsunterricht* when describing the English textbook by David Mair. In Germany, this teaching method aimed at the greatest possible self-driven activity of the student is mainly associated with the names of *Hugo Gaudig* and *Georg Kerschensteiner*. By *concentration* is meant the abandoning of unrelated, side by side teaching of the individual specialised disciplines without hand-in-hand cooperation and the introduction of deliberate joint working; ultimately, it means orienting the entire teaching to the educational goals of the school. The close relationship existing between the tendency of this present work and the demand for concentration has already previously been pointed out;[198] it was, however, firmly rejected its actual exaggerated expression in the Prussian school reform because of the impossibility of any appropriate teacher training.

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[197] Richtlinien für die Lehrpläne der höheren Schulen Preußens, Teil I und II, herausgegeben von Ministerialrat *Richert*. Berlin 1925. Weidmannsche Buchhandlung.

[198] Vol. I, pp. [301]–[302].

With respect to mathematics, the guidelines demand: "Between mathematics and other subjects as many connections as possible should be realised." For the movement to reform mathematics teaching, the idea of concentration in this form is nothing new; on the contrary, all reform objectives are a priori guided by it. Initially, it appears under the label Fusion requiring an appropriate link between the different disciplines of mathematics. Closely related is the insight that concepts central for all mathematics and penetrating it everywhere, such as those of function, of transformation and of group, must also be used to unify school mathematics. Finally, the demand made again and again by the reform movement to consider the applications of mathematics has exactly the same sense as the claim quoted above. Thus, it is understandable that mathematics teachers, if they want to adapt their teaching to the concentration requirement, can rely on an extensive literature. Apart from the reports of the German Subcommittee of IMUK mentioned in Volume I of the present work, of which in particular the third volume is devoted to the relations between mathematics and the neighbouring disciplines,[199] and the part edited by F. Klein of the series "Kultur der Gegenwart", published by Teubner,[200] which values the cultural importance of mathematics, the following publications should be mentioned here:

a) The *Habilitation* lecture by *Rudolf Schimmack:* "Uber die Verschmelzung verschiedener Zweige des mathematischen Unterrichts", printed in Heft 7 der 1. Folge der Berichte und Mitteilungen, veranlaßt durch die Internationale Mathematische Unterrichtskommision, Leipzig 1917.

b) The majority of the volumes of the series edited by *Walther Lietzmann* and *Alexander Witting mathematisch-physikalische Bibliothek* (Verlag Teubner, Leipzig). **[292]**

c) *Erich Salkowski*, "Der Gruppenbegriff als Ordnungsprinzip des geometrischen Unterrichts", Beiheft 7 of the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, Leipzig 1924.

d) *Georg Scheffers* und *Werner Kramer*, "Leitfaden der darstellenden und räumlichen Geometrie" I. Teil für Untertertia bis Untersekunda, Leipzig, Quelle & Meyer, 19.24, II. Teil für Obersekunda bis Oberprima, 1925.

This textbook is based on the view that for the development of the best possible space intuition, the fusion between planimetry and stereometry has to be dealt with more systematically and from an earlier time in school than has happened so far. If one starts to realise this idea of fusion, one encounters soon the necessity to perform spatial constructions graphically and to image solids on the plane. *The planimetry-stereometry-fusion urges therefore a broader notion of fusion, which comprises descriptive geometry*. However, according to Scheffers, orthographic projection and parallel perspective are relatively difficult to understand and can only be taught to more mature students. He thinks the vertical parallel projection applied to each topographical map is simple enough. In this method, each point is determined by its plan, i.e. by its projection to a plane thought of as horizontally and

[199] Already quoted in Vol. I, pp. [295]–[296].
[200] Already quoted in Vol. I, p. [305].

by a number that indicates the height of the point above the plane. This procedure is elaborated by Scheffers even more intuitively by indicating the heights not by numbers but by segments that can be taken from an accompanying drawing height scale. Of course, in the textbook, other methods of descriptive geometry are also presented. The authors of the new Prussian curriculum have adopted Scheffers' conception of and demand the teaching of the vertical parallel projection in *Untertertia* and *Obertertia*.

6. In the new curriculum, for the lower grades *Sexta* and *Quinta* a propaedeutic philosophical treatment of space forms is prescribed, for *Quarta* drawing the nets of simple bodies and their projection in a plane. The following two works will be of very great use to many teachers for this approach:

a) The book, based on rich pedagogical experience, by *Peter Treutlein:* "Der geometrische Anschauungsunterricht als Unterstufe eines zweistufigen geometrischen Unterrichts an unseren höheren Schulen", Leipzig 1911.

b) The highly interesting book, full of historical and cultural observations, by *Heinrich Emil Timerding* "Die Erziehung der Anschauung", Leipzig 1912.

[293] 7. The geometry textbook by *Peter Treutlein* and *Henrici* was mentioned in the report on p. [261]. Meanwhile, a great number of textbooks were published which consider the basic ideas of the reform movement. Among their great number, the following two should be mentioned:

a) *Otto Behrendsen* and *Eduard Götting*, Lehrbuch der Mathematik nach modernen Grundsätzen, Verlag Teubner, Leipzig, since 1908 in various editions.

b) Mathematisches Unterrichtswerk für höhere Knabenschulen, unter Mitwirkung von Paul B. Fischer, T. Zindler and Paul Zühlke, edited by *Walther Lietzmann*, Verlag Teubner, Leipzig. The part for the *Unterstufe* since 1916, that for the *Oberstufe* since 1920 in various editions.

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